8.323 Problem Set 7 Solutions

April 4th, 2023

Question 1: Momentum Conservation (10 points)

Consider an interacting field theory of a real scalar ϕ. Assume that the theory is translation invariant. Introduce the Fourier transform of

$$\begin{aligned} G_n(x_1, \dots, x_n) &= \langle \Omega \mathrm{T} \phi(x_1) \cdots \phi(x_n) | \Omega \rangle \\ \tilde{G}_n(p_1, \dots, p_n) &= \int d^4x_1 \cdots d^4x_n e^{-i \sum_i p_i \cdot x_i} G_n(x_1, \dots x_n) \end{aligned}$$

Show that

$$\tilde{G}_n(p_1, \dots, p_n) \propto (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n)$$

The idea is to use translation invariance to remove the x_1 dependence in G_n, so taking its Fourier transform gives a δ-function when integrating over x_1.
We compute:

$$\begin{aligned} \tilde{G}_n(p_1, \dots, p_n) &= \int d^4x_1 \cdots d^4x_n e^{-i \sum_{i=1}^n p_i \cdot x_i} G_n(x_1, x_2, \dots, x_n) \\ &= \int d^4x_1 d^4x_2 \cdots d^4x_n e^{-i \sum_{i=1}^n p_i \cdot x_i} G_n(0, x_2 - x_1, \dots, x_n - x_1) \\ &= \int d^4x_1 d^4x_2' \cdots d^4x_n' e^{-ix_1 \cdot p_1} e^{-i \sum_{i=2}^n (x_i' + x_1) \cdot p_i} G_n(0, x_2', \dots, x_n') \\ &= \int d^4x_2' \cdots d^4x_n' e^{-i \sum_{i=2}^n x_i' \cdot p_i} G(0, x_2', \dots, x_n') \int d^4x_1 e^{-ix_1 \cdot \sum_{i=1}^n p_i} \\ &= (2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n) \tilde{G}'(p_2, \dots p_n) \end{aligned}$$

In the second line we use translation invariance of the correlator. In the third line we change variables $x_i' = x_i - x_1$ for $i \geq 2$. In the last line we define $G'(x_2, \dots, x_n) := G(0, x_2, \dots, x_n)$ and $\tilde{G}'(p_2, \dots, p_n)$ its Fourier transform. We see that integrating over the x_1-dependence has given us the desired δ-function.

Question 2: Feynman Rules for a Complex Scalar Field (20 points)

For a complex scalar field particles and antiparticles are distinct, which we can think of as positively and negatively charged. The Feynman rules must distinguish them. We can make the distinction by including an arrow for each propagator indicating the flow of charge (note that this is separate from arrows indicating the flow of momentum, which are arbitrary). For a particle it is customary to point the arrow away from the external point in the initial state, and towards the external point in the final state. For an antiparticle, the direction is reversed. The charge arrows for propagators of the rest of a diagram then follow from charge conservation.

In order to avoid 2 kinds of arrows, we can simply align momentum arrows with the charge ones.

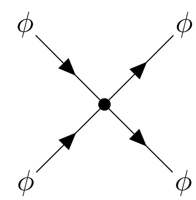
(a) Consider a complex scalar field theory

$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - \frac{\lambda}{4}(\phi^* \phi)^2$$

Write down the momentum space Feynman rules for this theory.

There are 2 Feynman rules: the propagator and a 4-point vertex. These correspond to the quadratic and 4th order terms in $\mathcal{L}$.

$$\frac{-i}{p^2 + m^2 - i\epsilon}$$



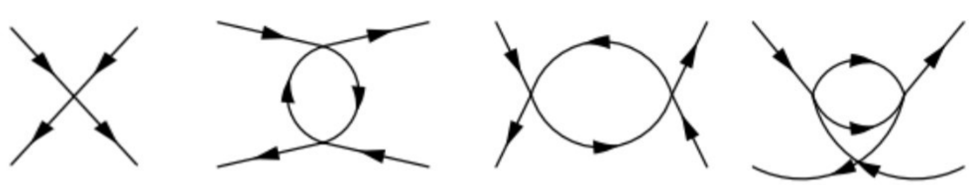
$$-i\lambda$$

(b) Draw the connected diagrams for the scattering amplitude for the process

$$\phi + \phi^* \to \phi + \phi^*$$

to order $\mathcal{O}(\lambda^2)$, where ϕ and ϕ^* denote particle and antiparticle.

We have 1 diagram at order λ. The rest are at order λ^2, corresponding to s, t, u-channel scattering. Note that we also have order λ^2 diagrams coming from corrections to the propagator, with loops on any of the 4 external legs, but these can be dropped since only amputated diagrams contribute to scattering amplitudes.

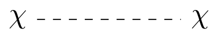


(c) Now suppose ϕ interacts with a real field χ via

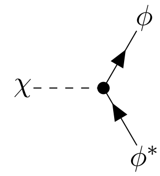
$$\mathcal{L}_I = \lambda' \chi \partial^\mu \phi^* \partial_\mu \phi$$

That is, the full Lagrangian is the sum of the free (ϕ, ϕ^*) theory, the free χ theory, and $\mathcal{L}_I$. Suppose χ has mass M. Use a solid line for the ϕ propagator and a dashed line for the χ propagator. Write down the momentum space Feynman rules for the full theory.

We write down the new Feynman rules, for the χ propagator and the $\chi\phi^*\phi$ interaction: In the second



$$\frac{-i}{p^2 + M^2 - i\epsilon}$$



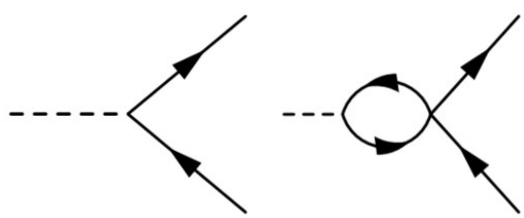
$$-i\lambda' p_1 \cdot p_2$$

diagram, the particles ϕ and ϕ^* have momentum p_1 and p_2, respectively. The Feynman rule comes from writing $i\mathcal{L}_I$ is momentum space, giving us the factor $i\lambda'(ip_1 \cdot p_2) = -i\lambda' p_1 \cdot p_2$.

(d) Suppose both $\lambda, \lambda' \sim \mathcal{O}(\epsilon)$, with ϵ a small parameter. Draw all the connected diagrams to order $\mathcal{O}(\epsilon^2)$ for the amplitude of the decay process

$$\chi \to \phi + \phi^*$$

We have 1 diagram at order ϵ given by the 3-point vertex. The rest at order ϵ^2 come from corrections to the external ϕ-propagators (which do not contribute to the decay process), and the 1-loop correction to the 3-point vertex (shown below).



Question 3: Higgs Production at LHC (10 points + 10 bonus)
The LHC collides protons with protons at very high energies. Each proton contains a number of quarks and gluons, so collisions of protons can also be considered as collisions of gluons. Here we consider a baby version of the Standard Model, which contains 3 types of scalar fields:

- Real gluon field g. Use a wavy line to denote its propagator.
- Complex quark field q. Use a solid line with an arrow to denote its propagator.
- Real Higgs field H. Use a dashed line to denote its propagator.

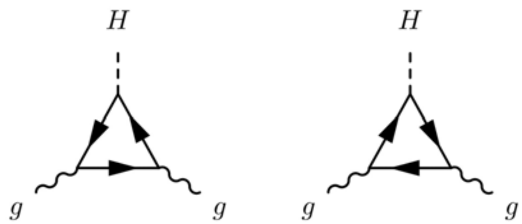
Suppose the interaction part of the theory is given by

$$\mathcal{L} = \lambda_1 g q^\dagger q + \lambda_2 H q^\dagger q$$

Note that there is no direct coupling between the gluon g and Higgs H. Assume that the couplings λ_1, λ_2 are small and of copmarable strength.
(a) The dominant channel of Higgs production at the LHC is gluon fusion, schematically written as

$$g + g \to H$$

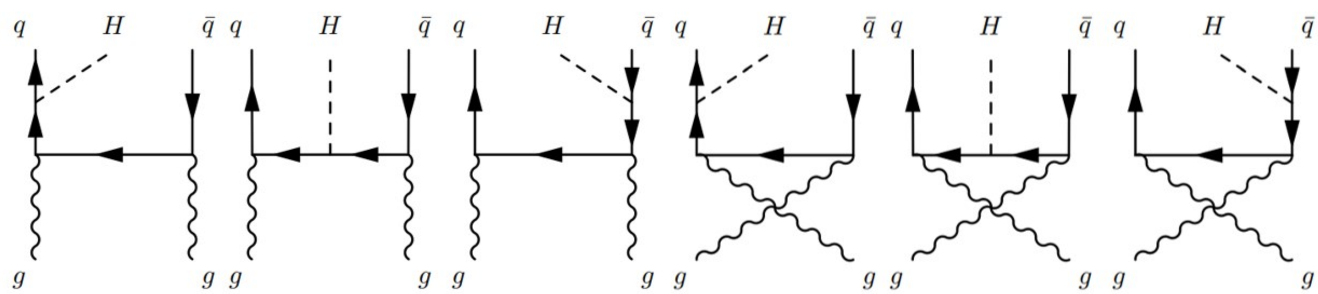
Draw the leading Feynman diagram(s) for this process.
The leading diagrams contribute at order $\lambda_1^2 \lambda_2$, and are given by the 2 'triangle diagrams' below. These 2 diagrams give equal contributions in our case, but with realistic vector gluons and fermionic quarks, they do not.



(b) **Bonus.** Another channel for Higgs production is

$$g + g \to H + q + \bar{q}$$

where $\bar{q}$ denotes antiquark. Draw the leading Feynman diagram(s) for this process.
The leading diagrams contribute at order $\lambda_1^2 \lambda_2$, and there are 6 of them given below.



Question 4: Properties of Gamma Matrices (25 points)

Without resorting to a particular representation, prove the following identities.

(a) $\mathrm{Tr}\,\gamma^\mu = 0$

The canonical way to do this is define the matrix $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$. Using $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we immediately have $\{\gamma^\mu, \gamma^5\} = 0$ and $\gamma_5^2 = 1$.

Now, we compute

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma^\mu\gamma^5\gamma^5 = -\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^5 = -\mathrm{Tr}\,\gamma^\mu\gamma^5\gamma^5 = -\mathrm{Tr}\,\gamma^\mu$$

In the third equality we use that γ^5 anticommutes with γ^μ, and in the 4th line we use the cyclicity of the trace. Therefore, $\mathrm{Tr}\,\gamma^\mu = 0$.

Note that the same argument goes through with $\gamma^5\gamma^5$ replaced by some $\gamma_\rho\gamma^\rho$ (no sum over ρ), with $\rho \neq \mu$.

(b) $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$

We compute

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \frac{1}{2}\mathrm{Tr}\left(\{\gamma^\nu, \gamma^\mu\}\right) = \frac{1}{2}\mathrm{Tr}\,2\eta^{\mu\nu}\mathbb{1} = 4\eta^{\mu\nu}$$

In the second equality we use the cyclicity of the trace.

(c) $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$

Again we use the γ^5 trick in (a):

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = \mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5 = (-1)^3\mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5 = -\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^5\gamma^5 = -\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda$$

Therefore $\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$. Note that the same argument goes through with $\gamma^5\gamma^5$ replaced by some $\gamma_\rho\gamma^\rho$ (no sum over ρ), with $\rho \neq \mu, \nu, \lambda$

(d) $\not{p}\not{q} = 2p\cdot q - \not{q}\not{p} = p\cdot q - 2i\Sigma^{\mu\nu}p_\mu q_\nu$, where we define $\not{p} = p_\mu\gamma^\nu$, and $\Sigma^{\mu\nu} = \dfrac{i}{4}[\gamma^\mu, \gamma^\nu]$

The first equality follows from contracting both sides of the anticommutation relation $\gamma^\mu\gamma^\nu = 2\eta^{\mu\nu} - \gamma^\nu\gamma^\mu$ with $p_\mu q_\nu$. To show the second equality, we use

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = \eta^{\mu\nu} - 2i\Sigma^{\mu\nu}$$

Contracting both sides by $p_\mu q_\nu$ gives the desired result.

(e) $\gamma^\mu\not{p}\gamma_\mu = -2\not{p}$

First note that

$$\gamma^\mu\gamma_\mu = \eta_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}\eta_{\mu\nu}\{\gamma^\mu, \gamma^\nu\} = \eta_{\mu\nu}\eta^{\mu\nu}\mathbb{1} = 4\mathbb{1}$$

We thus compute

$$\gamma^\mu\gamma^\nu\gamma_\mu = -\gamma^\mu\gamma_\mu\gamma^\nu + 2\gamma^\mu{\eta_\mu}^\nu = -4\gamma^\nu + 2\gamma^\nu = -2\gamma^\nu$$

Contracting with p_ν gives the desired result.

Question 5: Conserved 'Probability' Current of the Driac Equation (15 points)
Starting from the Dirac equation

$$(\gamma^\mu \partial_\mu - m)\psi = 0$$

(a) Show that one can construct a current j^μ which is conserved,

$$\partial_\mu j^\mu = 0$$

Note that taking the adjoint of the Dirac equation gives

$$0 = \psi^\dagger((\gamma^\mu)^\dagger(-\overleftarrow{\partial_\mu}) - m) = \psi^\dagger(\overleftarrow{\partial_\mu}\gamma^0\gamma^0(\gamma^\mu)^\dagger) - m\gamma^0) = -\bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu + m)\gamma^0$$

In the first equality we use that the derivative $\partial_\mu = -ip_\mu$ is anti-Hermitian, since the momentum is Hermitian. In the second equality we use $(\gamma^0)^2 = -1$. In the third equality we use $(\gamma)^\mu = \gamma^0\gamma^\mu\gamma^0$. Therefore $\bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$.

Now consider the object $j^\mu := \bar{\psi}\gamma^\mu\psi$, where $\bar{\psi} := \psi^\dagger\gamma^0$. We compute

$$\partial_\mu j^\mu = \bar{\psi}(\overleftarrow{\partial_\mu}\gamma^\mu)\psi + \bar{\psi}(\gamma^\mu\overrightarrow{\partial_\mu})\psi = -m\bar{\psi}\psi + m\bar{\psi}\psi = 0$$

(b) Show that the j^μ you constructed is real.
We take the conjugate:

$$(j^\mu)^* = (\bar{\psi}\gamma^\mu\psi)^* = \psi^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger\psi = \psi^\dagger(-(\gamma^0)^2)(\gamma^\mu)^\dagger(-\gamma^0)\psi = \bar{\psi}\gamma^0(\gamma^\mu)^\dagger\gamma^0\psi = \bar{\psi}\gamma^\mu\psi = j^\mu$$

Hence our current is real. In the 3rd equality we used that $(\gamma^0)^\dagger = \gamma^0\gamma^0\gamma^0 = -\gamma^0$.

(c) Show that by choosing the overall sign of j^μ, the zeroth component of j^μ

$$\rho := j^0$$

can be made to be postive definite.
We compute

$$\rho = j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = -\psi^\dagger\psi$$

This is negative definite. Thus by defining $j^\mu := -\bar{\psi}\gamma^\mu\psi$, we have $\rho = j^0 = \psi^\dagger\psi$ positive definite.