8.323 Problem Set 3 Solutions

February 28, 2023

Question 1: Lorentz Transformations for Operators and States (15 points)

(a) From the commutation relation of creation and annihilation operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

and the result of Problem 2(b) of Problem Set 1, argue that $a_{\mathbf{k}}$, $a^\dagger_{\mathbf{k}}$ should transform under a Lorentz transformation as

$$a_{\mathbf{k}} \to \tilde{a}_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad a^\dagger_{\mathbf{k}} \to \tilde{a}^\dagger_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a^\dagger_{\Lambda\mathbf{k}}$$

We start with the commutator

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')$$

The left-hand side transforms to $[\tilde{a}_{\mathbf{k}}, \tilde{a}^\dagger_{\mathbf{k}'}]$. Furthermore, since $\omega_{\mathbf{k}}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ is Lorentz-invariant (Problem 2 of PS1), the right-hand side transforms to

$$\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}(2\pi)^3\delta^{(3)}(\Lambda\mathbf{k}-\Lambda\mathbf{k}') = \frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}[a_{\Lambda\mathbf{k}}, a^\dagger_{\Lambda\mathbf{k}'}]$$

The transformations of the two sides must be equal, which is satisfied by imposing

$$a_{\mathbf{k}} \to \tilde{a}_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a_{\Lambda\mathbf{k}}, \qquad a^\dagger_{\mathbf{k}} \to \tilde{a}^\dagger_{\mathbf{k}} = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}} a^\dagger_{\Lambda\mathbf{k}}$$

(b) **Bonus.** Using the expression of $M_{\mu\nu}$ of Problem 4(c) of Problem Set 2, compute

$$\frac{1}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}], \qquad \frac{1}{2}[\omega_{\mu\nu}M^{\mu\nu}, a^\dagger_{\mathbf{k}}]$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal constants. Show that this indeed generates an infinitesimal Lorentz transformation in $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ which is consistent with part (a).

From Problem Set 2 we quote the result

$$M^{\mu\nu} = -\frac{i}{2}\int đ^3k\, k^\mu \left(a^\dagger_{\mathbf{k}}(\partial_{k_\nu} a_{\mathbf{k}}) - (\partial_{k_\nu} a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right) - (\mu \leftrightarrow \nu), \qquad \partial_{k_0} a_{\mathbf{k}} = 0, \qquad k^0 = \omega_{\mathbf{k}}$$

By antisymmetry, we have

$$\omega_{\mu\nu}M^{\mu\nu} = -i\int đ^3k\, k^\mu \left(a^\dagger_{\mathbf{k}}(\partial_{k_\nu} a_{\mathbf{k}}) - (\partial_{k_\nu} a^\dagger_{\mathbf{k}})a_{\mathbf{k}}\right)$$

Now we are ready to compute

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = -i\omega_{\mu\nu}\int đ^3k' k'^{\mu}\left([a^{\dagger}_{\mathbf{k}'}, a_{\mathbf{k}}](\partial_{k'_{\nu}} a_{\mathbf{k}'}) - [\partial_{\mathbf{k}'_{\nu}} a^{\dagger}_{\mathbf{k}'}, a_{\mathbf{k}}]a_{\mathbf{k}'}\right)\delta_{\nu\neq 0}$$
$$= i\omega_{\mu\nu}\int đ^3k' k'^{\mu}\left([a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}](\partial_{k'_{\nu}} a_{\mathbf{k}'}) - [a_{\mathbf{k}}, \partial_{\mathbf{k}'_{\nu}} a^{\dagger}_{\mathbf{k}'}]a_{\mathbf{k}'}\right)\delta_{\nu\neq 0}$$

We append the $\delta_{\nu\neq 0}$ for convenience because the integrand is zero when $\nu = 0$ (as we have used the convenient shorthand $\delta_{k_0} a_{\mathbf{k}} := 0$). This does nothing at the moment, but it will prevent confusion when we use the ladder-operator commutators, which will give factors like $\partial_{k'^0}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$. Moving on,

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = i\omega_{0i}\int đ^3k' \omega_{\mathbf{k}'}\left(\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_i} a_{\mathbf{k}'}) - (\partial_{k'_i}\delta^{(3)}(\mathbf{k}-\mathbf{k}'))a_{\mathbf{k}'}\right)$$
$$+ i\omega_{ij}\int đ^3k' k'^i\left(\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_j} a_{\mathbf{k}'}) - (\partial_{k'_j}\delta^{(3)}(\mathbf{k}-\mathbf{k}'))a_{\mathbf{k}'}\right) = (\mathrm{I}) + (\mathrm{II})$$

Note that the $\delta_{\nu\neq 0}$ added in the last step kills any ω_{i0} term. We first compute the second term:

$$(\mathrm{II}) = i\omega_{ij}\int đ^3k' \left(k'^i\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_j} a_{\mathbf{k}'}) + \partial_{k'_j}(k'^i a_{\mathbf{k}'})\delta^{(3)}(\mathbf{k}-\mathbf{k}')\right)$$
$$= i\omega_{ij}(k^i\partial_{k_j} a_{\mathbf{k}} + \partial_{k_j}(k^i a_{\mathbf{k}})) = 2i\omega_{ij}k^i\partial_{k_j} a_{\mathbf{k}} + i\omega_{ij}\delta^{ij}a_{\mathbf{k}} = 2i\omega_{ij}k^i\partial_{k_j} a_{\mathbf{k}}$$

where in the last equality, the second term vanishes because ω_{ij} is antisymmetric, but δ^{ij} is symmetric. We also have the first term:

$$(\mathrm{I}) = i\omega_{0i}\int đ^3k' \left(\omega_{\mathbf{k}'}\delta^{(3)}(\mathbf{k}-\mathbf{k}')(\partial_{k'_i} a_{\mathbf{k}'}) + \partial_{k'_i}(\omega_{\mathbf{k}'} a_{\mathbf{k}'})\delta^{(3)}(\mathbf{k}-\mathbf{k}')\right)$$
$$= i\omega_{0i}\left(\omega_{\mathbf{k}}(\partial_{k_i} a_{\mathbf{k}}) + \partial_{k_i}(\omega_{\mathbf{k}} a_{\mathbf{k}})\right) = 2i\omega_{0i}\omega_{\mathbf{k}}\partial_{k_i} a_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a_{\mathbf{k}}$$

Adding these together gives (where again we define $\delta_{k_0} a_{\mathbf{k}} := 0$)

$$[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = 2i\omega_{\mu\nu}k^{\mu}\partial_{k_{\nu}} a_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a_{\mathbf{k}} \tag{1}$$

Similarly, one can show that

$$[\omega_{\mu\nu}M^{\mu\nu}, a^{\dagger}_{\mathbf{k}}] = 2i\omega_{\mu\nu}k^{\mu}\partial_{k_{\nu}} a^{\dagger}_{\mathbf{k}} + i\omega_{0i}\frac{k^i}{\omega_{\mathbf{k}}}a^{\dagger}_{\mathbf{k}} \tag{2}$$

Finally, we want to show that this generates an infinitesimal Lorentz transformation consistent with (a). From part (a), we have

$$U_{\Lambda}a_{\mathbf{k}}U^{\dagger}(\Lambda) = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a_{\Lambda\mathbf{k}}, \qquad U_{\Lambda}a^{\dagger}_{\mathbf{k}}U^{\dagger}(\Lambda) = \sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a^{\dagger}_{\Lambda\mathbf{k}}$$

We can expand these for infinitesimal transformations

$$U_{\Lambda} = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}, \qquad (\Lambda k)_i = k_i + \omega_i{}^{\nu}k_{\nu}, \qquad \omega_{\Lambda\mathbf{k}} = (\Lambda k)_0|_{k^0=\omega_{\mathbf{k}}} = \omega_{\mathbf{k}} - \omega_0{}^{\nu}k_{\nu}$$

Substituting these expressions into the ones above yield

$$a_{\mathbf{k}} + \frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = \left(1 - \frac{1}{2\omega_{\mathbf{k}}}\omega_0{}^{\nu}k_{\nu}\right)(a_{\mathbf{k}} + \omega_i{}^{\nu}k_{\nu}\partial_{k_i} a_{\mathbf{k}})$$
$$\frac{i}{2}[\omega_{\mu\nu}M^{\mu\nu}, a_{\mathbf{k}}] = \left(-\frac{1}{2\omega_{\mathbf{k}}}\omega_0{}^{\nu}k_{\nu} + \omega_i{}^{\nu}k_{\nu}\partial_{k_i}\right)a_{\mathbf{k}} = -\frac{1}{2}\left(2\omega_{\mu\nu}k^{\mu}\partial_{k_{\nu}} + \omega_0{}^{\nu}\frac{k_i}{\omega_{\mathbf{k}}}\right)a_{\mathbf{k}}$$

Multiplying both sides by $-2i$ gives precisely (1). Replacing all instances of $a_{\mathbf{k}}$ with $a^{\dagger}_{\mathbf{k}}$ in this calculation gives (2), as desired. Therefore, $M^{\mu\nu}$ generates an infinitesimal Lorentz transformation acting on $a_{\mathbf{k}}$ and $a^{\dagger}_{\mathbf{k}}$ via commutation.

(c) **Bonus.** Now consider a unitary operator generating finite Lorentz transformations,

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

where $\omega_{\mu\nu}$ are finite constants. Show that

$$U_\Lambda|0\rangle = |0\rangle$$

i.e. the vacuum is Lorentz invariant. Assuming the Lorentz transformations of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$ in part (a), show that

$$U_\Lambda|k\rangle = |\Lambda k\rangle$$

Note that the expression for $M^{\mu\nu}$ in part (b) in terms of ladder operators is already normal ordered, with a lowering operator to the right in each term. Therefore, $M^{\mu\nu}|0\rangle = 0$ and

$$U_\Lambda|0\rangle = \sum_{n\geq 0}\frac{1}{n!}\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)^n|0\rangle = 1|0\rangle = |0\rangle$$

where in the second equality, only the $n = 0$ term does not vanish.

We can also act U_Λ on a momentum eigenstate $|k\rangle = \sqrt{2\omega_{\mathbf{k}}}a_{\mathbf{k}}^\dagger|0\rangle$:

$$U_\Lambda|k\rangle = \sqrt{2\omega_{\mathbf{k}}}U_\Lambda a_{\mathbf{k}}^\dagger U_\Lambda^\dagger U_\Lambda|0\rangle = \sqrt{2\omega_{\mathbf{k}}}\sqrt{\frac{\omega_{\Lambda\mathbf{k}}}{\omega_{\mathbf{k}}}}a_{\Lambda\mathbf{k}}|0\rangle = |\Lambda k\rangle$$

as desired. In the second equality we use the result from (a).

Question 2: Spatially Localized States of a Scalar Particle (25 points)

In a quantum field theory there is no natural way to define a position eigenvector $|\mathbf{x}\rangle$, as $\mathbf{x}$ is now simply a label, not an operator. Also, there is a fundamental conflict: a perfectly localized state in space is not a Lorentz covariant concept, as it picks out a reference frame (we cannot perfectly localize in time at the same time, as simultaneity is relative).

In this problem we will make these abstract statements concrete. Consider states of the form

$$|\mathbf{r}, t\rangle_f := \int d^3\mathbf{k} f(\mathbf{k}) |k\rangle e^{-i\mathbf{k}\cdot\mathbf{r} + i\omega_{\mathbf{k}} t}$$

where $f(\mathbf{k})$ is a function to be determined, and $|k\rangle = \sqrt{2\omega_{\mathbf{k}}} a^\dagger_{\mathbf{k}}|0\rangle$ is the 1-particle state of momentum $\mathbf{k}$ and energy $\omega_{\mathbf{k}}$.

(a) Determine $f(\mathbf{k})$ by the condition of perfect localization,

$${}_f\langle \mathbf{r}_1, t|\mathbf{r}_2, t\rangle_f = \delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$$

We compute:

$$\begin{aligned}
{}_f\langle \mathbf{r}_1, t|\mathbf{r}_2, t\rangle_f &= \int d^3\mathbf{k} d^3\mathbf{k}' f(\mathbf{k}) f^*(\mathbf{k}') \langle k'|k\rangle e^{-i(\mathbf{k}\cdot\mathbf{r}_1 - \mathbf{k}'\cdot\mathbf{r}_2) + it(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})} \\
&= \int d^3\mathbf{k} d^3\mathbf{k}' f(\mathbf{k}) f^*(\mathbf{k}') e^{-i(\mathbf{k}\cdot\mathbf{r}_1 - \mathbf{k}'\cdot\mathbf{r}_2) + it(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})} 2\omega_{\mathbf{k}} (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \\
&= (2\pi)^3 \int d^3\mathbf{k}\ 2\omega_{\mathbf{k}} |f(\mathbf{k})|^2 e^{-i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}
\end{aligned}$$

We see that expression is equal to $\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2) = \int đ^3k e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$

$$|f(\mathbf{k})|^2 = \frac{1}{(2\pi)^6}\frac{1}{2\omega_{\mathbf{k}}}, \qquad f(\mathbf{k}) = \frac{1}{(2\pi)^3\sqrt{2\omega_{\mathbf{k}}}}$$

where in going from $|f|^2$ to f the phase is not fixed, but we set it to 1 for simplicity. Therefore, using the notation $r^\mu = (t, \mathbf{r})$ we have

$$|\mathbf{r}, t\rangle = \int đ^3k \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} |k\rangle e^{-ik\cdot r} \tag{3}$$

(b) By acting the unitary operator U_Λ for a Lorentz transformation on the result in part (a), show that $|\mathbf{r}, t\rangle$ is not Lorentz invariant, i.e.

$$U_\Lambda |\mathbf{r}, t\rangle_f \neq |\Lambda\mathbf{r}, \Lambda t\rangle_f$$

Proof by contradiction. Suppose that we had $U_\Lambda|\mathbf{r}, t\rangle_f = |\Lambda\mathbf{r}, \Lambda t\rangle_f$. Then,

$${}_f\langle \Lambda\mathbf{r}_2, \Lambda t|\Lambda\mathbf{r}_1, \Lambda t\rangle_f = {}_f\langle \mathbf{r}_2, t|U^\dagger U|\mathbf{r}_1, t\rangle_f = {}_f\langle \mathbf{r}_2, t|\mathbf{r}_1, t\rangle_f$$

By part (a), the left-hand side is $\delta^{(3)}(\Lambda\mathbf{r}_1 - \Lambda\mathbf{r}_2)$, while the right-hand side is $\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$. One has:

$$\delta^{(3)}(\Lambda\mathbf{r}_1 - \Lambda\mathbf{r}_2) = \frac{\omega_{\mathbf{k}}}{\omega_{\Lambda\mathbf{k}}}\delta^{(3)}(\mathbf{r}_1 - \mathbf{r}_2)$$

so clearly the left and right-hand sides cannot be equal for generic Λ (a boost will change $\omega_{\mathbf{k}}$). This gives us our desired contradiction, therefore $|\mathbf{r}, t\rangle$ cannot be Lorentz-invariant.

(c) With $f(\mathbf{k})$ given by (a), consider the overlap $C(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2) = {}_f\langle \mathbf{r}_2, t_2 | \mathbf{r}_1, t_1 \rangle_f$. Evaluate $C(\mathbf{r}, t)$ for a spacelike separation $|\mathbf{r}| > t$.

Suppose we interpret $|{}_f\langle \mathbf{r}_2, t_2 | \mathbf{r}_1, t_1 \rangle_f|^2$ as the probability for the particle originally at time $\mathbf{r}_1$ and time t_1 to transition to time t_2. Would the propagation be causal (confined to the forwards light-cone)?

Using (3), we immediately have

$$C(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2) = \int đ^3\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1)} e^{-i\omega_\mathbf{k}(t_2 - t_1)}$$

This can be slickly evaluated by writing it as the derivative of a Lorentz-invariant integral:

$$C(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2) = 2i\partial_{t_2 - t_1} \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1)} e^{-i\omega_\mathbf{k}(t_2 - t_1)}$$

The integral is now Lorentz-covariant, hence we can evaluate it any frame. We are interested in spacelike-separated points $(\mathbf{r}_1, t)$, $(\mathbf{r}_2, t)$, hence there exists a Lorentz transformation that makes these points simultaneous, i.e. $\Lambda t_1 = \Lambda t_2$. The integral can then be evaluated in spherical coordinates.

$$\begin{aligned}
\int \frac{đ^3k}{2\omega_\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1)} e^{-i\omega_\mathbf{k}(t_2 - t_1)} &= \int \frac{đ^3\Lambda\mathbf{k}}{2\omega_{\Lambda\mathbf{k}}} e^{i\Lambda\mathbf{k}\cdot(\Lambda\mathbf{r}_2 - \Lambda\mathbf{r}_1)} e^{-i\omega_{\Lambda\mathbf{k}}(\Lambda t_2 - \Lambda t_1)} = \int \frac{đ^3\mathbf{k}'}{2\omega_{\mathbf{k}'}} e^{i\mathbf{k}'\cdot(\Lambda\mathbf{r}_2 - \Lambda\mathbf{r}_1)} \\
&= \frac{1}{2(2\pi)^3} \int_0^\infty d|\mathbf{k}| \frac{|\mathbf{k}|^2}{\sqrt{|\mathbf{k}|^2 + m^2}} \int_0^\pi d\phi \int_0^{2\pi} d\theta \sin\theta e^{ik|\mathbf{r}_1 - \mathbf{r}_2|\cos\theta} \\
&= \frac{1}{2(2\pi)^2} \int_0^\infty dk \frac{k^2}{\sqrt{k^2 + m^2}} \frac{2\sin(k|\mathbf{r}_2 - \mathbf{r}_1|)}{k|\mathbf{r}_2 - \mathbf{r}_1|} \\
&= \frac{m}{4\pi^2|\mathbf{r}_2 - \mathbf{r}_1|} K_1(m|\mathbf{r}_2 - \mathbf{r}_1|)
\end{aligned}$$

where $K_1(x)$ is the modified Bessel function of the second kind.

To compute $C(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2)$, we need to take the $\partial_{t_1 - t_2}$ on this object. But this is a bit unclear: we have chosen a frame where $t_1 = t_2$. To restore the time-dependence of this object, we use the fact that it is a Lorentz-invariant. This means that our integral can only depend on $|r_2 - r_1| = \sqrt{-(t_2 - t_1)^2 + (\mathbf{r}_2 - \mathbf{r}_1)^2}$ (using the notation $r = (\mathbf{r}, t)$), which in the specified frame reduces to $|\mathbf{r}_1 - \mathbf{r}_2|$. Therefore,

$$C(\mathbf{r}_1 - \mathbf{r}_2, t_1 - t_2) = 2i\partial_{t_2 - t_1} \left[\frac{m}{4\pi^2|r_2 - r_1|} K_1(m|r_2 - r_1|) \right] = \frac{im^2}{2\pi^2} \frac{t_2 - t_1}{(r_1 - r_2)^2} K_2(m|r_2 - r_1|)$$

Interpreting $|{}_f\langle \mathbf{r}_2, t_2 | \mathbf{r}_1, t_1 \rangle_f|^2$ as a probability, we have:

$$|{}_f\langle \mathbf{r}_2, t_2 | \mathbf{r}_1, t_1 \rangle_f|^2 = \left| \frac{m^2}{2\pi^2} \frac{t_2 - t_1}{(r_1 - r_2)^2} K_2(m|r_2 - r_1|) \right|^2$$

This is non-zero, meaning that the propagator is non-zero for generic spacelike separated points, i.e. propagation is non-causal.

(d) Compare your resulting state $|\mathbf{r}, t\rangle_f$ in part (a) with the state

$$|x\rangle := \phi(x)|0\rangle$$

where $x = (\mathbf{x}, t)$. Are the states $|x\rangle$ perfectly localized? Show that $|x\rangle$ is Lorentz covariant, i.e.

$$U_\Lambda |x\rangle = |\Lambda x\rangle$$

We compute

$$|x\rangle = \phi(x)|0\rangle = \int đ^3\mathbf{k}\frac{1}{\sqrt{2\omega_\mathbf{k}}}e^{-ik\cdot x}a^\dagger_\mathbf{k}|0\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-ik\cdot x}|k\rangle$$

Note that this last expression is written entirely of objects transforming simply under Lorentz transformations. The equal-time overlap is

$$\begin{aligned}
\langle x|x'\rangle &= \int đ^3\mathbf{k}đ^3\mathbf{k}'\frac{1}{2\omega_\mathbf{k}2\omega_{\mathbf{k}'}}e^{ik\cdot x - ik'\cdot x}\langle k'|k\rangle \\
&= \int đ^3\mathbf{k}đ^3\mathbf{k}'\frac{1}{2\omega_\mathbf{k}2\omega_{\mathbf{k}'}}e^{ik\cdot x - ik'\cdot x}2\omega_\mathbf{k}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}') \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \frac{m}{2\pi^2|\mathbf{x}-\mathbf{x}'|}K_1(m|\mathbf{x}-\mathbf{x}')
\end{aligned}$$

The integral in the second last equality has already been performed in part (c). The final expression is not equal to $\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, which would be the result if the states $|x\rangle$ were perfectly localized.

Lastly, we check Lorentz invariance.

$$U_\Lambda|x\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-ik\cdot x}U_\Lambda|k\rangle = \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}}e^{-ik\cdot x}|\Lambda k\rangle = \int \frac{đ^3\Lambda\mathbf{k}}{2\omega_{\Lambda\mathbf{k}}}e^{-i\Lambda k\cdot\Lambda x}|\Lambda k\rangle = |\Lambda x\rangle$$

(e) Consider a state

$$|\Psi\rangle = \int đ^3\mathbf{k}h(\mathbf{k})|k\rangle$$

with the corresponding 'wavefunction' defined as

$$\Psi(x) = \langle 0|\phi(x)|\Psi(x)\rangle$$

Find $h(\mathbf{k})$ so that at $t=0$, the single-particle wavefunction $\Psi(\mathbf{x})$ corresponding to $|\Psi\rangle$ is a Gaussian wave-packet centered around $\mathbf{x}_0$, with width a and momentum $\mathbf{p}$.

Given $|\Psi\rangle$ defined this way, at $t=0$ its wavefunction is given by

$$\Psi(\mathbf{x}) = \langle 0|\phi(x)|\Psi\rangle = \int đ^3\mathbf{k}h(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

This is just a 3D-Fourier transform. Therefore, choosing

$$h(\mathbf{k}) = (2\pi^{1/2}a)^{3/2}e^{-\frac{a^2}{2}(\mathbf{k}-\mathbf{p})^2}e^{-i\mathbf{k}\cdot\mathbf{x}_0} \qquad \Longrightarrow \qquad \Psi(\mathbf{x}) = \frac{1}{(\pi^{1/2}a)^{3/2}}e^{-\frac{1}{2a^2}(\mathbf{x}-\mathbf{x}_0)^2}e^{i\mathbf{p}\cdot\mathbf{x}}$$

which is a Gaussian centered around $\mathbf{x}_0$, with width a and momentum $\mathbf{p}$.

Question 3: Normal Ordering and Smeared Fields (25 points)

Consider the free real scalar field theory discussed in lecture.

(a) First show that

$$\langle 0|\phi(\mathbf{x},t)|0\rangle = 0$$

Evaluate the vacuum expectation value

$$\sigma^2 := \langle 0|\phi^2(\mathbf{x},t)|0\rangle$$

Express σ^2 as an integral over a single variable, and show that the integral is divergent. This result signifies that the vacuum is not empty! While the expectation value of ϕ is zero, the fluctuations of ϕ, as measured by σ^2, are non-zero, and in fact are infinitely large. This is a reflecton of that a QFT has an infinite number of degrees of freedom.

The expectation value of ϕ is

$$\langle 0|\phi(\mathbf{x},t)|0\rangle = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(\langle 0|a_\mathbf{k}|0\rangle e^{ik\cdot x} + \langle 0|a^\dagger_\mathbf{k}|0\rangle e^{-ik\cdot x}\right) = 0 + 0 = 0$$

Where we have used $a_\mathbf{k}|0\rangle = 0$, and $\langle 0|a^\dagger_\mathbf{k} = 0$.

The variance of ϕ has 4 sets of terms when we take the mode expansion, corresponding to combinations of creation/annihilation operators. The only set of terms that does not vanish is $\langle 0|a_\mathbf{k}a^\dagger_{\mathbf{k}'}|0\rangle$, hence:

$$\begin{aligned}
\sigma^2 = \langle 0|\phi(\mathbf{x},t)|0\rangle &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a_\mathbf{k}a^\dagger_{\mathbf{k}'}|0\rangle e^{i(k-k')\cdot x} \\
&= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(k-k')\cdot x} \\
&= \int \frac{đ^3\mathbf{k}}{2\omega_\mathbf{k}} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{k^2+m^2} = \infty
\end{aligned}$$

(b) The general philosophy of QFT is to regard operators like $\phi^2(\mathbf{x},t)$ as 'bad' operators. One then can introduce 'good' operators which do not suffer divergences, a procedure referred to as renormalization. One way to remove the divergence in part (a) is to introduce normal-ordered operators. The rule of normal ordering is whenever one has products of $a_\mathbf{k}$'s and $a^\dagger_\mathbf{k}$'s, to move all the $a_\mathbf{k}$'s to the right of the $a^\dagger_\mathbf{k}$'s. We denote the normal ordered version of an operator $\mathcal{O}$ by $:\mathcal{O}:$. For example,

$$:a_{\mathbf{k_1}}a^\dagger_{\mathbf{k_2}}a_{\mathbf{k_3}}a^\dagger_{\mathbf{k_4}}:= a^\dagger_{\mathbf{k_2}}a^\dagger_{\mathbf{k_4}}a_{\mathbf{k_1}}a_{\mathbf{k_3}}$$

Express the normal-ordered operator $:\phi^2(\mathbf{x},t):$ in terms of $\mathbf{a_k}$ and $\mathbf{a}^\dagger_\mathbf{k}$. Show $\langle 0|\, :\phi^2(\mathbf{x},t):\, |0\rangle = 0$, and

$$\phi^2(\mathbf{x},t) =:\phi^2(\mathbf{x},t): +\sigma^2\mathbb{1}$$

The mode expansion for $\phi^2(\mathbf{x},t)$ is given by

$$\phi^2(\mathbf{x},t) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a_\mathbf{k}a_\mathbf{k}e^{i(k+k')\cdot x} + a^\dagger_\mathbf{k}a^\dagger_\mathbf{k}e^{-i(k+k')\cdot x} + a_\mathbf{k}a^\dagger_\mathbf{k}e^{i(k-k')\cdot x} + a^\dagger_\mathbf{k}a_\mathbf{k}e^{-i(k-k')\cdot x}\right)$$

Of the 4 sets of terms composing the integrand, only the third is not normal ordered. Therefore,

$$:\phi^2(\mathbf{x},t):= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left(a_\mathbf{k}a_\mathbf{k}e^{i(k+k')\cdot x} + a^\dagger_\mathbf{k}a^\dagger_\mathbf{k}e^{-i(k+k')\cdot x} + a^\dagger_\mathbf{k}a_\mathbf{k}e^{i(k-k')\cdot x} + a^\dagger_\mathbf{k}a_\mathbf{k}e^{-i(k-k')\cdot x}\right)$$

Observe that each term in this expression has a lowering operator to the right, or a raising operator to the left. Therefore, when acting on the vaccum we have that $\langle 0|\,{:}\phi^2(\mathbf{x},t){:}\,|0\rangle = 0$.

Lastly we have seem that ${:}\phi^2(\mathbf{x},t){:}$ differs from the original $\phi^2(\mathbf{x},t)$ by just a single commutator:

$$\begin{aligned}\phi^2(\mathbf{x},t) - {:}\phi^2(\mathbf{x},t){:} &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}]e^{i(k-k')\cdot x} \\ &= \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')e^{i(k-k')\cdot x} = \sigma\mathbb{1}\end{aligned}$$

where in the last line we quote the result from (a).

(c) Another way to introduce 'good' operators is to consider the 'smeared' field

$$\tilde{\phi}(\mathbf{x},t) := N\int d^3y\phi(\mathbf{y},t)e^{-\frac{1}{a^2}(\mathbf{x}-\mathbf{y})^2}, \qquad N = \frac{1}{\pi^{3/2}a^3}$$

The definition is motivated from the fact that the divergence in part (a) comes from having 2 ϕ's at the same spacetime point. Here we thus smear ϕ in a region of radius a. N is a normalization factor, chosen so that

$$\lim_{a\to 0}\tilde{\phi}(\mathbf{x},t) = \phi(\mathbf{x},t)$$

Show that

$$\langle 0|\tilde{\phi}(\mathbf{x},t)|0\rangle = 0$$

Now consider the fluctuations of $\tilde{\phi}$,

$$\tilde{\sigma}^2 := \langle 0|\tilde{\phi}^2(\mathbf{x},t)|0\rangle$$

Express $\tilde{\sigma}^2$ as an integral over a single variable, and show that it is finite.

We first compute the expectation, using the result from (a) that $\langle 0|\phi(\mathbf{y},t)|0\rangle = 0$.

$$\langle 0|\tilde{\phi}(\mathbf{x},t)|0\rangle = N\int d^3y e^{-\frac{1}{a^2}(\mathbf{x}-\mathbf{y})^2}\langle 0|\phi(\mathbf{y},t)|0\rangle = 0$$

Next, the variance.

$$\begin{aligned}\tilde{\sigma}^2 := \langle 0|\tilde{\phi}^2(\mathbf{x},t)|0\rangle &= N^2\int d^3\mathbf{y}d^3\mathbf{z}\frac{đ^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{đ^3\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\langle 0|a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}|0\rangle e^{-(\mathbf{x}-\mathbf{y})^2/a^2}e^{-(\mathbf{x}-\mathbf{z})^2/a^2}e^{i\mathbf{k}\cdot\mathbf{y}-i\mathbf{k}'\cdot\mathbf{z}} \\ &= N^2\int đ^3\mathbf{k}d^3\mathbf{y}d^3\mathbf{z}\frac{1}{2\omega_{\mathbf{k}}}e^{-(\mathbf{x}-\mathbf{y})^2/a^2}e^{-(\mathbf{x}-\mathbf{z})^2/a^2}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{z})} \\ &= \int\frac{đ^3\mathbf{k}}{2\omega_{\mathbf{k}}}e^{-\frac{1}{2}a^2\mathbf{k}^2} = \frac{4\pi}{2(2\pi)^3}\int_0^\infty dk\frac{k^2}{\sqrt{k^2+m^2}}e^{-\frac{1}{2}a^2k^2}\end{aligned}$$

This last integral can be evaluated in terms of hypergeometric or Bessel functions. Note to marker: student gets full credit if the integral is not evaluated. We find

$$\tilde{\sigma}^2 = \frac{1}{8a^2\pi^{3/2}}U\left(\frac{1}{2},0,\frac{a^2m^2}{2}\right) = \frac{m^2}{16\pi^2}e^{a^2m^2/4}\left(K_1(\tfrac{a^2m^2}{4}) - K_0(\tfrac{a^2m^2}{4})\right)$$

(d) Without evaluated the integral in part (c), show that in the limits of small and large a, the leading term in $\tilde{\sigma}^2$ may be written as

$$\tilde{\sigma}^2 \approx \alpha a^\gamma$$

Compute α and γ for each of these 2 limits. One should discover that at large a the average field approaches a classical variable, whereas at small a it is dominated by fluctuations.

We seek to compute the integral

$$\tilde{\sigma}^2 = \frac{1}{4\pi^2}\int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} e^{-\frac{1}{2}a^2k^2} = \frac{1}{4\pi^2 a^2}\int_0^\infty du \frac{u^2}{\sqrt{u^2+m^2a^2}} e^{-u^2/2}$$

First we expand for small a. The prefactor to the exponential can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{u\left(1+\frac{m^2a^2}{2u^2}+\cdots\right)} \approx u$$

Hence, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2 a^2}\int_0^\infty du u e^{-u^2/2} = \frac{1}{4\pi^2 a^2}$$

For large a, the prefactor to the exponential can be expanded as:

$$\frac{u^2}{\sqrt{u^2+m^2a^2}} = \frac{u^2}{ma\left(1+\frac{u^2}{2m^2a^2}+\cdots\right)} \approx \frac{u^2}{ma}$$

Hence, the integral

$$\tilde{\sigma}^2 \approx \frac{1}{4\pi^2 m a^3}\int_0^\infty du u^2 e^{-u^2/2} = \frac{1}{\sqrt{32\pi^3} m a^3}$$

We see that $\tilde{\sigma}^2 \to \infty$ for small a, meaning that the field is dominated by fluctuations. On the other hand, $\tilde{\sigma}^2 \to 0$ for large a, meaning that the field behaves like a classical variable.

Question 4: Correlation Functions for a Complex Scalar (15 points)
Consider a theory of a complex scalar field ϕ which is invariant under a phase rotation of ϕ. The unitary operator U_α generating a phase rotation is

$$U_\alpha = e^{-i\alpha Q}, \qquad Q = i\int d^3x(\pi_{\phi^*}\phi^* - \pi_\phi\phi)$$

We also assume that the vacuum of the theory is invariant under the phase rotation,

$$U_\alpha|0\rangle = |0\rangle$$

The theory can be interacting. That is, in this problem, you should not use the mode expansion for ϕ discussed in lecture, which applies only to a free field theory.
(a) Show that

$$U_\alpha\phi(x)U_\alpha^\dagger = e^{i\alpha}\phi(x), \qquad U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi^*(x)$$

First consider the case where Q and $\phi(x)$ are evaluated at the same time t. Using Baker-Campbell-Hausdorff we compute

$$\begin{aligned}U_\alpha\phi(x)U_\alpha^\dagger = \phi(x) + \alpha\int d^3x'[-\phi(x')\pi_\phi(x'), \phi(x)]\\ + \frac{\alpha^2}{2!}\int d^3x'd^3x''[-\phi(x'')\pi_\phi(x''), [-\phi(x')\pi_\phi(x'), \phi(x)]] + \cdots\end{aligned}$$

Note that $[-\phi(x')\pi_\phi(x'), \phi(x)] = i\phi(x')\delta^{(3)}(\mathbf{x}-\mathbf{x}')$, whose $\int d^3\mathbf{x}$-integral yields $i\phi(x)$. Therefore, doing this process n times for the n-th order term yields $i^n\phi(x)$, and

$$U_\alpha\phi(x)U_\alpha^\dagger = \left(1 + i\alpha + \cdots + \frac{(i\alpha)^n}{n!} + \cdots\right)\phi(x) = e^{i\alpha}\phi(x)$$

If instead, Q is computed at some time $t' = x'^0$ which is different from $t = x^0$, we can time-evolve $\phi(x)$ to the timeslice t', and use that $[H, Q] = 0$:

$$\begin{aligned}U_\alpha\phi(\mathbf{x}, t)U_\alpha^\dagger &= e^{-i\alpha Q(t')}e^{iH(t-t')}\phi(\mathbf{x}, t')e^{-iH(t-t')}e^{i\alpha Q(t')}\\ &= e^{iH(t-t')}e^{-i\alpha Q(t')}\phi(\mathbf{x}, t')e^{i\alpha Q(t')}e^{-iH(t-t')}\\ &= e^{iH(t-t')}e^{i\alpha}\phi(\mathbf{x}, t')e^{-iH(t-t')} = e^{i\alpha}\phi(\mathbf{x}, t)\end{aligned}$$

where in the 3rd equality we quote the result we have just shown above.
An almost identical calculation yields

$$U_\alpha\phi^*(x)U_\alpha^\dagger = e^{-i\alpha}\phi(x)$$

(b) Show that

$$\langle 0|\phi(x)\phi(x')|0\rangle = 0$$

Using the result of part (a), along with the invariance of the vacuum, we have

$$\langle 0|\phi(x)\phi(x')|0\rangle = \langle U_\alpha 0|\phi(x)U_\alpha^\dagger U_\alpha\phi(x')U_\alpha^\dagger|0\rangle = e^{2i\alpha}\langle 0|\phi(x)\phi(x')|0\rangle$$

If we choose α not a multiple of 2π, then the equality is not satisfied unless $\langle 0|\phi(x)\phi(x')|0\rangle = 0$.

(c) Now consider a general n-point function of products of ϕ's and ϕ^*'s inserted at different spacetime points between the vacuum, i.e

$$\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle$$

Show that this vanishes whenever the numbers of ϕ's and ϕ^*'s are not the same.

Consider a general n-point function with M instances of ϕ and N instances of ϕ^*', with $n = M + N$. Similarly to (b), we have

$$\begin{aligned}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi_n(x_n)|0\rangle &= \langle 0|U_\alpha\phi(x_1)U_\alpha^\dagger U_\alpha\cdots U_\alpha^\dagger U_\alpha\phi^*(x_i)U_\alpha^\dagger U_\alpha\cdots U_\alpha^\dagger U_\alpha\phi_(x_n)U_\alpha^\dagger|0\rangle \\ &= e^{i(M-N)\alpha}\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle\end{aligned}$$

By picking again α arbitrary (e.g. irrational), we see that the equality is not satisfied for $M \neq N$ unless $\langle 0|\phi(x_1)\cdots\phi^*(x_i)\cdots\phi(x_n)|0\rangle = 0$.

That is, the n-point function may only be non-vanishing is the numbers of ϕ's and ϕ^*'s are the same.

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