Quantum Field Theory I (8.323) Spring 2023
Assignment 10

Apr. 18, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3
- Peskin & Schroeder Chap. 9.5
- Peskin & Schroeder Chap. 4.7

Notes: conventions and some useful formulae

1. Conventions of γ matrices:
$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$
and
$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form
$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$
and the action is given by
$$S = -i\int d^4x\,\bar{\psi}(\not{\partial} - m)\psi \ . \tag{4}$$

3. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

4. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as
$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{5}$$
$u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal
$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{6}$$

5. With normalization (5), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{7}$$

and the orthogonal relations (6) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{8}$$

These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (5).

6. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}^{(s)} u_s(\vec{k}) e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k}) e^{-ik\cdot x} \right] \ . \tag{9}$$

where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}'), \tag{10}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{11}$$

Problem Set 10

1. **Chiral symmetry (15 points)**

   Consider the Dirac action with $m = 0$.

   (a) Show that the action is invariant under transformations

   $$\psi \to e^{i\alpha\gamma^5}\psi \ . \tag{12}$$

   (b) Construct the Noether current for the above symmetry.

   (c) Find how the mass term $m\bar{\psi}\psi$ transforms under (12). Is it invariant?

2. **Quantizing the theory of Majorana fermions (25 points)**

   Consider the theory of Majorana fermions discussed in Pset 9, written in terms of a two-component complex spinor ψ_L

   $$\mathcal{L}_L = i\psi_L^\dagger\sigma^\mu\partial_\mu\psi_L - \frac{m}{2}(\psi_L^T\sigma^2\psi_L + \psi_L^\dagger\sigma^2\psi_L^*). \tag{13}$$

   where ψ^T denotes the transpose of ψ, and $\sigma^\mu = (1, \vec{\sigma})$.

(a) Write down the equal time canonical quantization relations.

(b) Write down the classical equations of motion. In momentum space a general solution can be written as

$$\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x} \,. \tag{14}$$

Using the above notations, write down a complete basis of solutions in the rest frame (i.e. $\vec{p} = 0$).

(c) Verify the following expressions give a complete basis of solutions for general p (below $\bar{\sigma} = (1, -\vec{\sigma})$)

$$\begin{aligned} u_s(p) &= \sqrt{-p\cdot\bar{\sigma}}\zeta_s & (15)\\ v_s(p) &= -\sqrt{-p\cdot\bar{\sigma}}\sigma^2\zeta_s & (16) \end{aligned}$$

where $\zeta_s, s = \pm$ are respectively eigenvectors of σ^3 with eigenvalues $\pm$.

(d) Write down the mode expansion for quantum operator ψ_L.

(e) Define the vacuum and construct the single-particle states (properly normalized). Discuss the differences between the particles in this theory and those of the Dirac theory.

3. **Gaussian integrals for Grassman variables (16 points)**

Show that

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i)\ e^{-\theta_i^* A_{ij}\theta_j} = \det A \tag{17}$$

and

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i)\ \theta_k\theta_l^* e^{-\theta_i^* A_{ij}\theta_j} = \det A (A^{-1})_{kl} \,. \tag{18}$$

4. **Yukawa theory (24 points)**

Consider the Yukawa theory discussed in lecture

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - i\bar{\psi}(\not{\partial} - m)\psi - g\phi\bar{\psi}\psi \,. \tag{19}$$

Denote the propagator of a ϕ particle by a dashed line and that of ψ by a solid line (with arrow). We will call p the particle excitation of ψ and $\bar{p}$ and the anti-particle excitation of ψ.

(a) Consider the process

$$\bar{p} + \bar{p} \to \bar{p} + \bar{p} \tag{20}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), (p_2, s_2)$ and $(p_1', s_1'), (p_2', s_2')$ respectively.

(b) Consider the process

$$p + \bar{p} \to \phi + \phi \tag{21}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), (p_2, s_2)$ and p_1', p_2' respectively.

(c) Consider the process

$$p + \phi \to p + \phi \tag{22}$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and ploarizations $(p_1, s_1), p_2$ and $(p_1', s_1'), p_2'$ respectively.

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