Quantum Field Theory I (8.323) Spring 2023
Assignment 5

Mar. 7, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.4
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. Path integral is a subject covered by graduate quantum mechanics class. In lectures I tried to give a self-contained exposition of the essential ideas, but have not time to be comprehensive. If you have not done path integral before you should consult standard quantum mechanics textbooks on this subject. A very nice, and detailed discussion of the formalism with many examples and applications is the classic book by *Feynman and Hibbs "Quantum Mechanics and Path Integrals."*

   The problems in this pset are aimed to remind you of the subject (or familiarize it if you have not done it before).

2. For a non-relativistic particle moving in one-dimension with a Hamiltonian

$$H = \frac{p^2}{2m} + V(x) \tag{1}$$

   we have, in the path integral formalism,

$$\begin{aligned} & K(x_a, t_a; x_b, t_b) = \langle x_a, t_a | x_b, t_b \rangle & (2) \\ = \; & \lim_{N \to \infty} \left(\frac{m}{2\pi i \Delta t} \right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dx_1 \cdots dx_{N-1} \; \exp \left[i \Delta t \sum_{i=0}^{N-1} \left(\frac{m}{2} \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 - V(x_i) \right) \right] & \\ \equiv \; & \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) \; \exp \left[i \int_{t'}^{t} dt \, L(\dot{x}, x) \right] & (3) \end{aligned}$$

   where $L = \frac{1}{2} m \dot{x}^2 - V(x)$ is the Lagrangian and $N \Delta t = t_a - t_b$.

Problem Set 5

1. **A useful formula and path integral in phase space (20 points)**

   (a) Derive equations (4) and (5) below.

$$
\begin{aligned}
&\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle \\
&= \int \frac{dp_i}{2\pi} \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i)\right] && (4)\\
&= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im\Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right] && (5)
\end{aligned}
$$

   Equation (5) was used in lecture to derive (3).

   (b) Using (4) to derive an alternative expression of (2):

$$
\langle x_a, t_a | x_b, t_b \rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t)\ \exp\left[i \int_{t'}^{t} dt\, (p\dot{x} - H)\right] . \tag{6}
$$

   The integration $\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t)$ in the above expression should be understood as

$$
\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t) \equiv \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \int \frac{dx_1 dp_1}{2\pi} \cdots \int \frac{dx_{N-1} dp_{N-1}}{2\pi} \tag{7}
$$

   where we again divide the interval $[t', t]$ into N segments with $t_0 = t', t_N = t$.

2. **Schrodinger equation rederived (20 points)**

   Using the propagator introduced in (3), the wave function $\psi(x, t)$ for a system at time t can be obtained from that at time t' by

$$
\psi(t, x) = \int dx'\, K(x, t; x', t')\, \psi(x', t') . \tag{8}
$$

   Show that $\psi(t, x)$ satisfies the Schrodinger equation

$$
i\partial_t \psi(t, x) = -\frac{1}{2m}\partial_x^2 \psi(t, x) + V(x)\psi(t, x) . \tag{9}
$$

   [Hint:] We can write the wave function at time $t + \delta t$ from that at time t by:

$$
\psi(t + \delta t, x) = \int dy\, K(t + \delta t, x; t, y)\, \psi(t, y) . \tag{10}
$$

With δt small, $K(t+\delta t, x; t, y)$ can be written as a single infinitesimal step of (3), i.e.

$$K(t+\delta t, x; t, y) = \left(\frac{m}{2\pi i \delta t}\right)^{\frac{1}{2}} \exp\left[i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)\right] . \tag{11}$$

Proving (9) then amounts to a careful analysis of the $\delta t \to 0$ limit of the difference between (10) and $\psi(t, x)$.

3. **Free particle (20 points)**

   For a free particle, i.e. $V(x) = 0$, perform explicitly the integrals over $x_i, i = 1, \cdots N-1$ in (3) and show that

$$K(x_a, t_a; x_b, t_b) = \left(\frac{m}{2\pi i(t_a - t_b)}\right)^{\frac{1}{2}} \exp\left[\frac{im(x_a - x_b)^2}{2(t_a - t_b)}\right] . \tag{12}$$

   [**Hint:**] First do the integral for x_1 and try to find a pattern.

4. **Path integral for a free particle revisited (20 points)**

   In this problem we evaluate the path integral for a free particle using a different method from Prob. 3. For simplicity we take $x_a = x_b = 0$ and $t_b = 0$ and $t_a = T$. As discussed in lecture K is a Gaussian path integral of the form

$$K(0, T; 0, 0) = \int_{x(0)=0}^{x(T)=0} Dx(t) \exp\left[\frac{i}{2}\int dtdt'\, x(t)A(t,t')x(t')\right] \tag{13}$$

   for some differential operator A.

   (a) Write down the explicit expression for A.

   (b) Find all the eigenvalues of A. Show that the determinant of A can be written as

$$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2} . \tag{14}$$

   (c) Since (13) is Gaussian, it can be evaluated as

$$K(0, T; 0, 0) = \frac{C}{\sqrt{\det A}} \tag{15}$$

   where C is some constant. By comparing the above equation with (12) show that consistency of the two approaches requires

$$\frac{C}{\sqrt{\det A}} = \left(\frac{m}{2\pi i T}\right)^{\frac{1}{2}} \tag{16}$$

i.e.

$$C \prod_{n=1}^{\infty} \frac{T}{\sqrt{m}\pi} \frac{1}{n} = \left(\frac{m}{2\pi i T} \right)^{\frac{1}{2}} \ . \tag{17}$$

Note: While the above equation looks rather bizarre, it can in fact be shown explicitly by discretizing the path integral as in the second line of (3), carefully calculating C, and then taking $N \to \infty$ (after discretizing, A becomes a finite matrix).

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