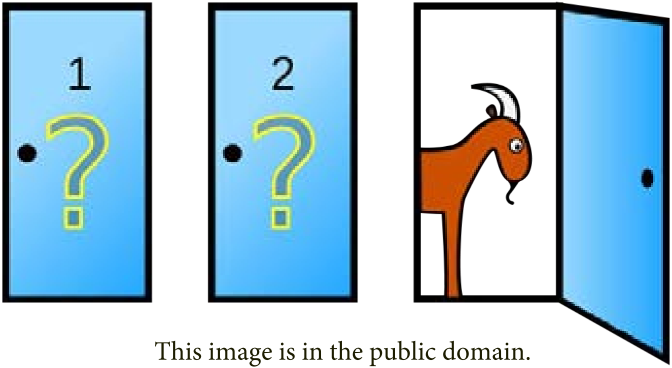
Conditional Probability, Independence, Bayes' Theorem
18.05 Spring 2022



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Announcements/Agenda

Announcements

- Slides and problems are posted before class. Solutions right after class.
- Make use of office hours.
- Solution code is posted on MITx.

Agenda

- Studio 1 comments
- Conditional probability
- Multiplication rule; Law of total probability
- Bayes' Theorem
- Base rate fallacy

Studio 1

- Really well done! About $1/2$ the class did the optional problem.
- Take-away: Simulation is an (often) easy way to estimate probabilities.
- In 2(b), the exact probability for $n = 23$ is slightly more than 0.5. We accepted 23 or 24 as answers.
- Comment out extra print or cat statements.
- Print just what is asked for, plus comments to the grader.
- Instructions for finding your graded code is on MITx (right side of page, under Course Handouts)

Sample Space Confusions

1. Sample space = *set* of all possible outcomes of an experiment.
2. The size of the set is **NOT** the sample space.
3. Outcomes can be sequences of numbers.

Examples.

1. Roll 5 dice: $S =$ set of all sequences of 5 numbers between 1 and 6, e.g. $(1, 2, 1, 3, 1, 5) \in S$.
The size $|S| = 6^5$ is not a set.

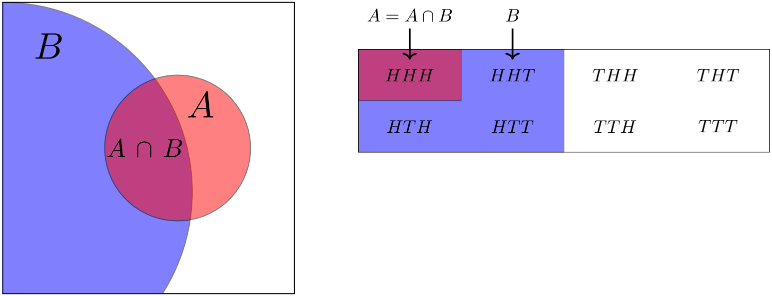
2. $S =$ set of all sequences of 10 birthdays,
e.g. $(111, 231, 3, 44, 55, 129, 345, 14, 24, 14) \in S$.
$|S| = 365^{10}$

3. n some number, $S =$ set of all sequences of n birthdays.
$|S| = 365^n$.

Conditional Probability

'the probability of A given B'.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ provided } P(B) \neq 0.$$



Conditional probability: Abstractly and for coin example

Table/Concept Question

(Discuss with your table and then click in your answer.)

Toss a coin 4 times.
Let $A =$ 'at least three heads'
and $B =$ 'first toss is tails'.

1. What is $P(A|B)$?

   (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

2. What is $P(B|A)$?

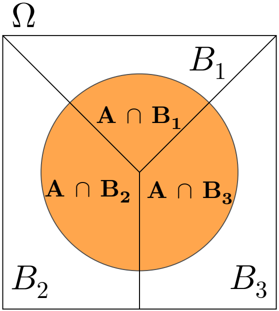
   (a) 1/16 (b) 1/8 (c) 1/4 (d) 1/5

Multiplication Rule, Law of Total Probability

Multiplication rule: $P(A \cap B) = P(A|B) \cdot P(B)$.

Law of total probability: If B_1, B_2, B_3 *partition* S then

$$\begin{aligned} P(A) &= P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) \\ &= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3) \end{aligned}$$

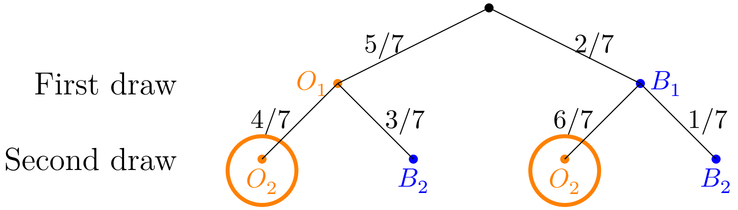


Trees

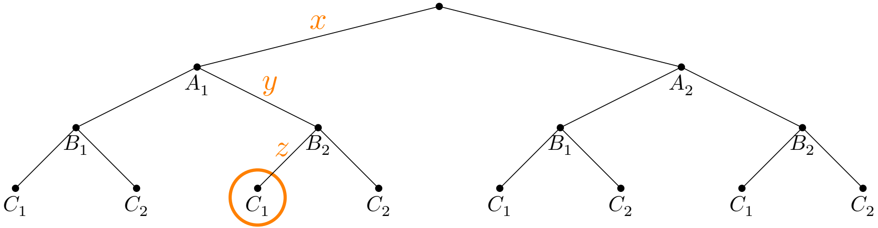
- Organize computations
- Compute total probability
- Compute Bayes' formula

Example. Game: 5 orange and 2 blue balls in an urn. A random ball is selected and replaced by a ball of the other color; then a second ball is drawn.

1. What is the probability the second ball is orange?
2. What is the probability the first ball was orange given the second ball was orange?



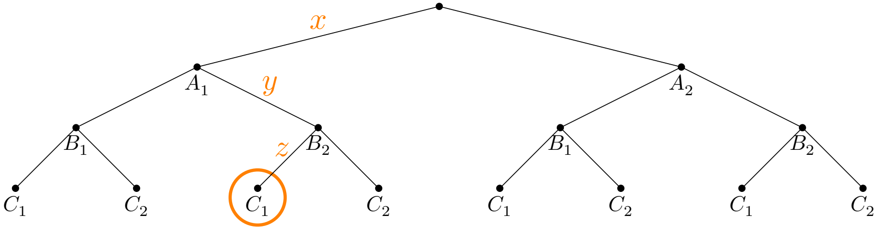
Concept (clicker) Question: Trees 1



1. The probability x represents

(a) $P(A_1)$
(b) $P(A_1|B_2)$
(c) $P(B_2|A_1)$
(d) $P(C_1|B_2 \cap A_1)$.

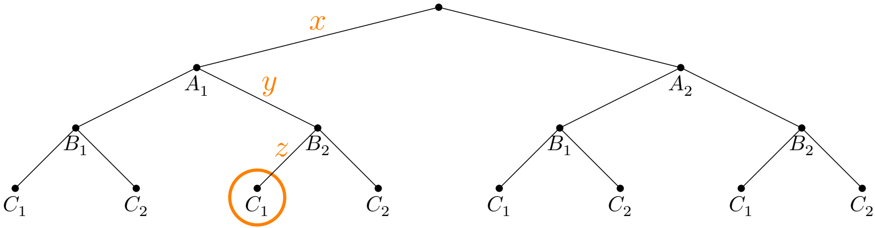
Concept (clicker) Question: Trees 2



2. The probability y represents

(a) $P(B_2)$
(b) $P(A_1|B_2)$
(c) $P(B_2|A_1)$
(d) $P(C_1|B_2 \cap A_1)$.

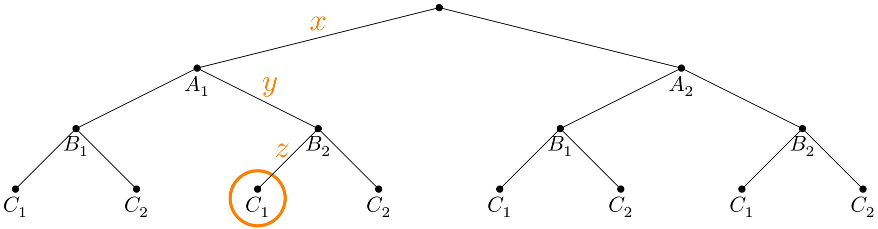
Concept Question: Trees 3



3. The probability z represents

(a) $P(C_1)$
(b) $P(B_2|C_1)$
(c) $P(C_1|B_2)$
(d) $P(C_1|B_2 \cap A_1)$.

Concept Question: Trees 4



4. The circled node represents the event

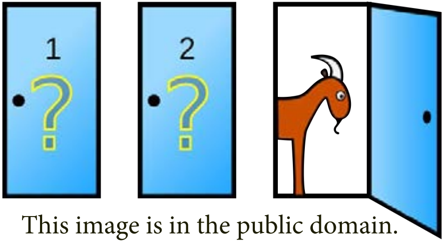
(a) C_1
(b) $B_2 \cap C_1$
(c) $A_1 \cap B_2 \cap C_1$
(d) $C_1 | B_2 \cap A_1$.

Let's Make a Deal with Monty Hall

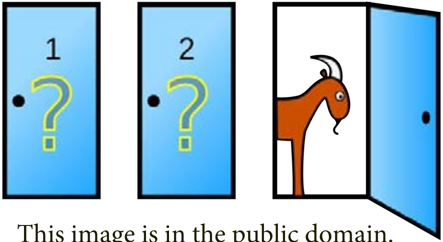
- One door hides a car, two hide goats.
- The contestant chooses any door.
- Monty always opens a different door with a goat. (He can do this because he knows where the car is.)
- The contestant is then allowed to switch doors if they want.

What is the best strategy for winning a car?

(a) Switch (b) Don't switch (c) It doesn't matter



Board question: Monty Hall



Organize the Monty Hall problem into a tree and compute the probability of winning if you always switch.

Hint first break the game into a sequence of actions.

Independence

Events A and B are independent if the probability that one occurred is not affected by knowledge that the other occurred.

$$\begin{aligned} \text{Independence} &\Leftrightarrow P(A|B) = P(A) \quad (\text{provided } P(B) \neq 0) \\ &\Leftrightarrow P(B|A) = P(B) \quad (\text{provided } P(A) \neq 0) \end{aligned}$$

(For any A and B)

$$\Leftrightarrow P(A \cap B) = P(A)P(B)$$

Table Question: Independence

Roll two dice and consider the following events

- $A =$ 'first die is 3'
- $B =$ 'sum is 6'
- $C =$ 'sum is 7'

A is independent of

(a) B and C (b) B alone

(c) C alone (d) Neither B or C.

Bayes' Theorem

Also called Bayes' Rule and Bayes' Formula.

Allows you to find $P(A|B)$ from $P(B|A)$, i.e. to 'invert' conditional probabilities.

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Often compute the denominator $P(B)$ using the law of total probability.

Board Question: Evil Squirrels

Of the one million squirrels on MIT's campus most are good-natured. But one hundred of them are pure evil! An enterprising student in Course 6 develops an "Evil Squirrel Alarm" which they offer to sell to MIT for a passing grade. MIT decides to test the reliability of the alarm by conducting trials.

Evil Squirrels Continued

1000000 squirrels, 100 are evil.

- When presented with an evil squirrel, the alarm goes off 99% of the time.
- When presented with a good-natured squirrel, the alarm goes off 1% of the time.

(a) If a squirrel sets off the alarm, what is the probability that it is evil?
(b) Should MIT co-opt the patent rights and employ the system?

One solution
(This is a base rate fallacy problem)

We are given:

$$P(\text{nice}) = 0.9999, \qquad P(\text{evil}) = 0.0001 \text{ (base rate)}$$

$$P(\text{alarm} \mid \text{nice}) = 0.01, \quad P(\text{alarm} \mid \text{evil}) = 0.99$$

$$P(\text{evil} \mid \text{alarm}) = \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm})}$$

$$= \frac{P(\text{alarm} \mid \text{evil}) P(\text{evil})}{P(\text{alarm} \mid \text{evil}) P(\text{evil}) + P(\text{alarm} \mid \text{nice}) P(\text{nice})}$$

$$= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.01)(0.9999)}$$

$$\approx 0.01$$

Squirrels continued

Summary:

$$\text{Probability a random test is correct} = 0.99$$

$$\text{Probability a positive test is correct} \approx 0.01$$

These probabilities are not the same!

Alternative method of calculation:

	Evil	Nice	
Alarm	99	9999	10098
No alarm	1	989901	989902
	100	999900	1000000

Board Question: Dice Game

1. The Randomizer holds the 6-sided die in one fist and the 8-sided die in the other.
2. The Roller selects one of the Randomizer's fists and covertly takes the die.
3. The Roller rolls the die in secret and reports the result to the table.

Given the reported number, what is the probability that the 6-sided die was chosen? (Find the probability for each possible reported number.)