Recitation 7

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1 Representations of the Lorentz Group

The Lorentz Group

The Lorentz algebra $\mathfrak{so}(1,3)$ is generated by the set of infinitesimal boosts and rotations. Equivalently, it can be defined by the commutation relations obeyed by the generators $(\mathcal{J}^{\lambda\rho})^{\mu}{}_{\nu}$:

$$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\rho\sigma}] = i(\eta^{\alpha\sigma}\mathcal{J}^{\beta\rho} - \eta^{\alpha\rho}\mathcal{J}^{\beta\sigma} - \eta^{\beta\sigma}\mathcal{J}^{\alpha\rho} + \eta^{\beta\rho}\mathcal{J}^{\alpha\sigma})$$

To get an element of the Lorentz group, we can exponentiate some linear combination of these generators.

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right)$$

The Lorentz algebra is an example of what we call a Lie algebra, defined by some generators, their linear combinations, and a commutator or 'Lie bracket'. The Lorentz group is the exponential of the algebra, or equivalently the algebra is the set of infinitesimal elements of the group.

Representations

A representation of the Lorentz algebra is a map from $\mathfrak{so}(1,3)$ into matrices in a way that preserves the central commutation relation above. It provides an explicit realization of the algebra into a set of matrices that we can manipulate. More formally, a representation is a map $d : \mathfrak{so}(1,3) \to \mathfrak{gl}(V)$ such that

$$[d(\mathcal{J}^{\alpha\beta}), d(\mathcal{J}^{\rho\sigma})] = d([\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\rho\sigma}])$$

Here $\mathfrak{gl}(V)$ is the space of linear transformations of a vector space V, itself equivalent to the space of matrices acting on V. To give a representation then, is to prescribe the vector space V along with the action of the generators (i.e. their matrix forms). Sometimes their matrix forms are clear from context, so we denote a representation by just V.

Any representation of the algebra $\mathfrak{so}(3,1)$ induces a representation $D : \mathrm{SO}(1,3) \to \mathrm{GL}(V)$ of the Lorentz group by exponentiating:

$$D(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\rho\sigma}d(\mathcal{J}^{\rho\sigma})\right), \qquad D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2)$$

Here $\mathrm{GL}(n)$ is the space of invertible matrices. The proof of the second equation amounts to the Baker-Campbell-Hausdorff formula.

Field Representations

This holds for general representations. Now we can talk about the representations formed by the set of some fields $\{\Phi_a(x)\} = V$. The action of the Lorentz group is given by

$$D(\Lambda)^{ab}\Psi^b(\Lambda^{-1}x)$$

Another way of saying this is that this is how our field transforms under a Lorentz transformation. The transformation of the spacetime location is given by the inverse of the transformation, which should be familiar from working with scalar fields. We also transform the internal indices, which depend on the representation that the field lives in: examples include scalars, vectors, and tensors, which are given by

$$\phi(x) \to \phi(\Lambda^{-1}x), \qquad A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x), \qquad T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}(\Lambda^{-1}x)$$

The scalar has 1 component, the vector has 4 components, and the tensor has 16. For this reason, we often denote these representations by **1**, **4**, **16**. Comparing this to the general equation, we have $D_{\mathbf{1}}(\Lambda) = 1$, $D_{\mathbf{4}}(\Lambda) = \Lambda^\mu{}_\nu$, $D_{\mathbf{16}}(\Lambda) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta$.

Irreducible Representations

Not all representations are created equal: some are simpler than others, and serve as 'building blocks' from which we can construct all the representations we care about. To talk about these, we first introduce the notion of an invariant subspace W, or ideal, of a representation V as one that is mapped to itself under all Lorentz transformations. That is,

$$D(\Lambda)W = \{D(\Lambda)w, w \in W\} \subseteq W$$

If $W \subseteq V$ is invariant, it's not hard to show that $W^\perp$ is also invariant, and thus one can decompose V into a direct sum:

$$V = W \oplus W^\perp$$

If each subspace is invariant, the action of the Lorentz group acts independently on each subspace, with no mixing. Viewing the $D(\Lambda)$'s as matrices acting on V, this means that we can block diagonalize all the matrices at once, as off-diagonal terms correspond to mixing between W and $W^\perp$. Obviously V and $\emptyset$ are always invariant, these are the trivial invariant subspaces of V.

A representation is irreducible (irrep.) if there are no non-trivial invariant subspaces. That is, we cannot block diagonalize all $D(\Lambda)$'s at once, meaning one cannot factor V into smaller building block representations. In this sense, you may think of irreps as prime numbers. Let us see whether our familiar scalar, vector, and 2-tensor field representations are irreducible.

1. Scalars. All Lorentz transformations act trivially as the identity on a dimension 1 object, the matrix is just (1). There are no non-trivial subspaces, so **1** is an irrep.
2. Vectors. This is the representation **4**. It is irreducible, as one cannot simultaneously block diagonalize the rotations and boosts. It is called the fundamental representation.
3. Tensors. This is the representation **16**. It has 3 invariant subspaces.
   1: The scalar subspace, given by span$\{\eta^{\mu\nu}\}$; this is left invariant under Lorentz by definition.

   $$(D_{\mathbf{16}}(\Lambda)\eta)^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$$

   6: The space of antisymmetric tensors of dimension 6, $B^{\mu\nu} = -B^{\nu\mu}$. One can check:

   $$(D_{\mathbf{16}}(\Lambda)B)^{\mu\nu} := \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\alpha\beta} = -\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta B^{\beta\alpha} = -(D_{\mathbf{16}}(\Lambda)B)^{\nu\mu}$$

   It turns out that the **6** decomposes into 2 invariant dimension 3 subspaces, $\mathbf{6} = \mathbf{3} \oplus \bar{\mathbf{3}}$
   Here the **3** and $\bar{\mathbf{3}}$ are self-dual and anti-self-dual tensors, i.e. $G^{\mu\nu} = \pm i\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}$.
   9: The space of symmetric tensors of dimension 10, $S^{\mu\nu} = S^{\nu\mu}$.
   In total, the decomposition of the tensor representation into irreps is thus

   $$\mathbf{16} = \mathbf{1} \oplus \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{9}$$

Our goal is to classify all the irreducible representations of the Lorentz group. once we do this, we can build any other representation we care about.

2 Interlude: SU(2)

Before doing this for the Lorentz group, let us first turn to a simpler example which you should all be familiar with, SU(2). This is the Lie group of 2×2 unitary matrices with unit determinant,

$$\mathrm{SU}(2) = \{U \in M_{2\times 2}(\mathbb{C}), \quad U^\dagger U = UU^\dagger = 1, \quad \det U = 1\}$$

The corresponding Lie algebra is $\mathfrak{su}(2)$. It has 3 generators, $\{J_1, J_2, J_3\}$ satisfying the relation

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

Exponentiating this algebra gives SU(2): any group element can be written as $U = e^{i\theta_i J_i}$. It's often easier to look for representations of the algebra instead of the group, so we will look for all irreducible representations of $\mathfrak{su}(2)$. Exponentiating these, we can recover all the irreducible representations of SU(2). That is, we look for a set of $r \times r$ matrices $d_s(J_i)$ satisfying the above commutation relation, that cannot be simultaneously block diagonalized.

The classification is simple. There is a unique $\mathfrak{su}(2)$ irrep for each dimension $r \geq 1$. These correspond to particles of different spins, which is made clear by defining $l = (r-1)/2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. In this context $r = 2l + 1$ is the number of orthogonal states a spin l particle can have, often labelled $m = -l, \ldots l$.

What are the $r \times r$ matrices that satisfy our commutation relation? For spin-0 the matrix is trivial, $d_0(J_1, J_2, J_3) = (0)$: angular momentum acts trivially. For spin-1/2, $r = 2$, and the matrices are given by the familiar Paulis. The objects on which these act form a vector space, and are called Pauli spinors.

$$d_{1/2}(J_1, J_2, J_3) = \frac{1}{2}(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0, & -1 \end{pmatrix}\right)$$

In $d \geq 3$, we have generalizations of the Pauli matrices. These are called Wigner D-matrices, and are given by

$$d_l(J_1) \propto \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad d_l(J_2) \propto \begin{pmatrix} 0 & -i & & \\ i & 0 & -i & \\ & i & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \qquad d_l(J_3) = \mathrm{diag}(-l, -1+1, \ldots, l-1, l)$$

To summarize: all irreducible representations of $\mathfrak{su}(2)$ are classified by an integer $l \geq 1$ representing its dimension. Exponentiating them gives representations of SU(2).

3 Classification of Irreducible Representations of the Lorentz Group

3.1 The Classification

Now we've built up enough foundations to perform our classification. This hinges on one fact which we will show–the Lorentz algebra looks like 2 copies of $\mathfrak{su}(2)$.

We often think of the Lorentz algebra $\mathfrak{so}(1,3)$ being generated by the rotations and boosts J_i and K_i. We will work in an alternative basis $\{J_i^\pm\}$, given by the generators

$$J_i^\pm := \frac{1}{2}(J_i \pm iK_i)$$

Using the commutation relations of rotations and boosts, we can compute the commutators in this basis:

$$[J_i^{\pm}, J_j^{\pm}] = i\epsilon_{ijk}J_k^{\pm}, \qquad [J_i^{\pm}, J_j^{\mp}] = 0$$

This form is particularly insightful, and should be familiar. Namely, we have 2 copies $\{J_i^+\}$ and $\{J_i^-\}$ of the $\mathfrak{su}(2)$ algebra, which are completely decoupled. We have thus shown the result

$$\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

How do we classify irreps of a direct product? Generally, we may have something like $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where we know the irreps of $\mathfrak{h}_{1,2}$ as $\{d_a^1, V_a\}$ and $\{d_b^2, W_b\}$. The irreps of the larger algebra is given by the tensor product: $\{d_a^1 \otimes d_b^2, V_a \otimes W_b\}$. Practically, this means that we take the tensor product of the vector spaces on which they act, and the tensor product of the matrices as well. The irreps of $\mathfrak{g}$ are then labelled by the pair (a, b).

Let us return to the Lorentz algebra. We know the irreps of $\mathfrak{su}(2)$ are labelled by a non-negative half-integer. The irreps of $\mathfrak{so}(1,3)$ then, are labelled by a pair (j_+, j_-), for $j_+, j_- \in \{0, \frac{1}{2}, 1, \ldots\}$. By the usual rules of the vector space tensor product, the dimension is given by the product of the j_+ and j_- dimensions:

$$\dim(d_{j_+,j_-}) = (2j_+ + 1)(2j_- + 1)$$

In this notation, we may now write the Lorentz scalar as $\mathbf{1} = (0,0)$, the vector as $\mathbf{4} = (\frac{1}{2}, \frac{1}{2})$, and the tensor as $\mathbf{16} = \mathbf{1} + \mathbf{3} + \bar{\mathbf{3}} + \mathbf{9} = (0,0) + (0,1) + (1,0) + (1,1)$.

If $j_+ + j_-$ is an integer we call this a tensor representation, and if it a half-integer we call it a spinor representation.

3.2 What is a Spinor?

The simplest spinor representations are $d_L = (\frac{1}{2}, 0)$ and $d_R = (0, \frac{1}{2})$. We call these the left and right-handed Weyl spinor representations, the reason we call them left and right will be explained next lecture. One transforms as a Pauli spinor under J_i^+ but as a scalar (trivially) under J_i^-, while the other does the opposite. That is,

$$d_L(J_i^+) = \tfrac{1}{2}\sigma, \quad d_L(J_i^-) = 0, \qquad d_L(J_i) = \frac{1}{2}\sigma_i, \quad d_L(K) = -\frac{i}{2}\sigma_i$$

Or, written in another way, the matrix representation of each generator is as follows.

$$d_L(\mathcal{J}^{\mu\nu}) = \frac{1}{2}\begin{pmatrix} 0 & -i\sigma_1 & -i\sigma_2 & -i\sigma_3 \\ i\sigma_1 & 0 & \sigma_3 & -\sigma_2 \\ i\sigma_2 & -\sigma_3 & 0 & \sigma_1 \\ i\sigma_3 & \sigma_2 & -\sigma_1 & 0 \end{pmatrix}$$

Both d_L and d_R are 2-dimensional representations. One describes left-handed spin-1/2 particles, while the other describes right-handed spin-1/2 particles. If we want particles that are not chiral, and also have a mass term, we can take their direct sum, which is called the Dirac representation.

$$s = d_L \oplus d_R = (\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})$$

Note that it is not irreducible, and has dimension $2 + 2 = 4$. That is, Dirac spinors are 4 component fields. Using the matrices above, it is straightforwards to show that for the Dirac representation,

$$d(\mathcal{J}^{\mu\nu}) = -\frac{i}{4}[\sigma^\mu, \sigma^\nu] = \Sigma^{\mu\nu}$$

which you should be familiar with from lecture. This consistency check shows that the bispinor representation describes the Dirac particles discussed in class. To obtain the irreps of the Lorentz group, we just exponentiate the irreps of the Lorentz algebra. For instance, for the Dirac representation we have:

$$S(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu} d(\mathcal{J}^{\mu\nu})} = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}$$

To conclude, we list out the low dimensional representations of the Lorentz group.

Representation	(j_+, j_-)	Dimension	Irrep?	(Beyond) SM Fields
Scalar	$(0,0)$	1	Y	Higgs, (Dilaton), (Axion)
Left Weyl	$(\frac{1}{2},0)$	2	Y	Left-handed Neutrinos
Right Weyl	$(0,\frac{1}{2})$	2	Y	(Right-handed Neutrinos)
Dirac/bispinor	$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$	4	N	Electrons, Quarks
Vector	$(\frac{1}{2},\frac{1}{2})$	4	N	Photon, Gluon, W, Z
Antisymmetric Tensor	$(1,0)\oplus(0,1)$	6	N	(Kalb-Ramond)
Rarita-Schwinger	$(1,\frac{1}{2})$, $(\frac{1}{2},1)$	6	Y	(Gravitino)
Traceless Symmetric Tensor	$(1,1)$	9	Y	(Graviton)

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