Theorem. Let m *and* n *be integers; let* $n > 0$. *Let*

$$h(y) = (\sqrt[n]{y})^m \quad \text{for} \quad y > 0.$$

Then h *is differentiable, and*

$$h'(y) = \frac{m}{n} (\sqrt[n]{y})^m \cdot y^{-1} .$$

Proof. Step 1. We first prove the theorem in the case $m = 1$. Let $f(x) = x^n$ for $x > 0$. Then the inverse function to f, denoted $g(y)$, is the n^{th} root function. By the theorem on the derivative of an inverse function, $g'(y)$ exists and

$$g'(y) = \frac{1}{f'(x)} ,$$

where $x = g(y)$. Now $f'(x) = n\,x^{n-1}$. Therefore

$$g'(y) = \frac{1}{n\,x^{n-1}} = \frac{1}{n(\sqrt[n]{y})^{n-1}}$$

$$= \frac{1}{n(\sqrt[n]{y})^n\,(\sqrt[n]{y})^{-1}} = \frac{\sqrt[n]{y}}{n\,y}$$

$$= \frac{1}{n}\sqrt[n]{y} \cdot y^{-1} .$$

Step 2. We prove the theorem in general. If $m = 0$, it is trivial. Otherwise, we apply the chain rule. We have

$$h(y) = (\sqrt[n]{y})^m ;$$

then

$$h'(y) = m(\sqrt[n]{y})^{m-1} \left[\frac{1}{n}\sqrt[n]{y} \cdot y^{-1}\right]$$

$$= \frac{m}{n} (\sqrt[n]{y})^m \cdot y^{-1}. \quad \square$$

Once one has checked that the laws of exponents hold for rational exponents (Notes G), one can write this formula in a manner that is much easier to remember:

<u>Theorem</u>. <u>Let</u> r <u>be a rational constant; let</u> $h(x) = x^r$ <u>for</u> $x > 0$. <u>Then</u> h <u>is differentiable and</u>

$$h'(x) = rx^{r-1}.$$

We will give a different proof of this theorem later on, one which holds when r is an arbitrary real constant.