

L'Hopital's rule for 0/0

Theorem. Suppose  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  as  
 $x \rightarrow a$ . If

$$\frac{f'(x)}{g'(x)} \rightarrow L \quad \text{as} \quad x \rightarrow a,$$

then also  $f(x)/g(x) \rightarrow L$  as  $x \rightarrow a$ .

This result holds whether  $a$  and  $L$  are finite or  
infinite, and it also holds if the limits are one-sided.

Proof. The proof when  $a$  is finite is that given on p. 295 of the text. The crucial step is to use Cauchy's mean-value theorem to prove that  $g(x) \neq 0$  for  $x$  near  $a$ , and that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

for some  $c$  between  $a$  and  $x$ . It follows that if  $f'(x)/g'(x)$  approaches  $L$  as  $x \rightarrow a$ , then  $f(x)/g(x)$  must approach  $L$  also. In the text, it is assumed that  $L$  is finite. But it does really not matter whether  $L$  is finite or infinite; precisely the same proof applies.

The proof in the case  $a = +\infty$  is given on p. 298 of the text. Again, it is assumed that  $L$  is finite, but that doesn't matter; if  $L$  is  $\pm \infty$  precisely the same proof applies.  $\square$

Remark. L'Hopital's rule also works if  $f(x)$  and  $g(x)$  both approach  $\infty$  instead of  $0$ . But the proof is more

complicated. We shall give a proof shortly. The only cases of interest to us concern the logarithm and the exponential. For these functions, a direct proof is given on p. 301 of the text. Alternatively, they may be treated by using L'Hopital's rule for the case  $\infty/\infty$ , as we shall see.

The behavior of  $\log$  and  $\exp$  that we are concerned with is stated in the following theorem:

Theorem. As  $x \rightarrow +\infty$ , both  $\log x$  and  $e^x$  approach  $+\infty$ . But  $\log x$  approaches  $\infty$  more slowly than any positive power of  $x$ , and  $e^x$  approaches  $\infty$  more rapidly than any positive power of  $x$ ; the same holds for any positive powers of  $\log x$  and  $e^x$ . More precisely, if  $a$  and  $b$  are positive real numbers, then

$$\lim_{x \rightarrow +\infty} \frac{(\log x)^b}{x^a} = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} \frac{(e^x)^b}{x^a} = +\infty$$

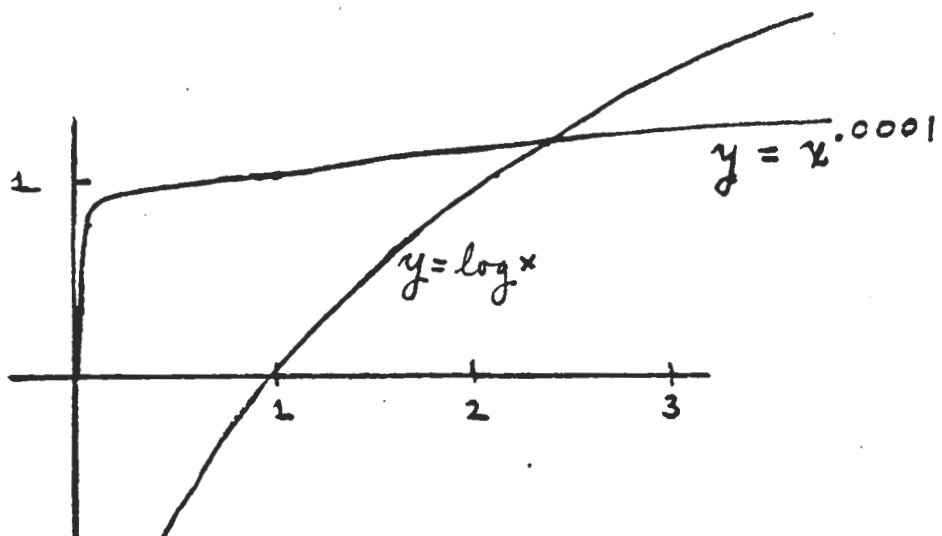
Corollary. The function  $\log x$  goes to  $-\infty$  very slowly as  $x$  goes to  $0$ . More precisely, if  $a$  is a positive real number, then

$$\lim_{x \rightarrow 0^+} x^a \log x = 0.$$

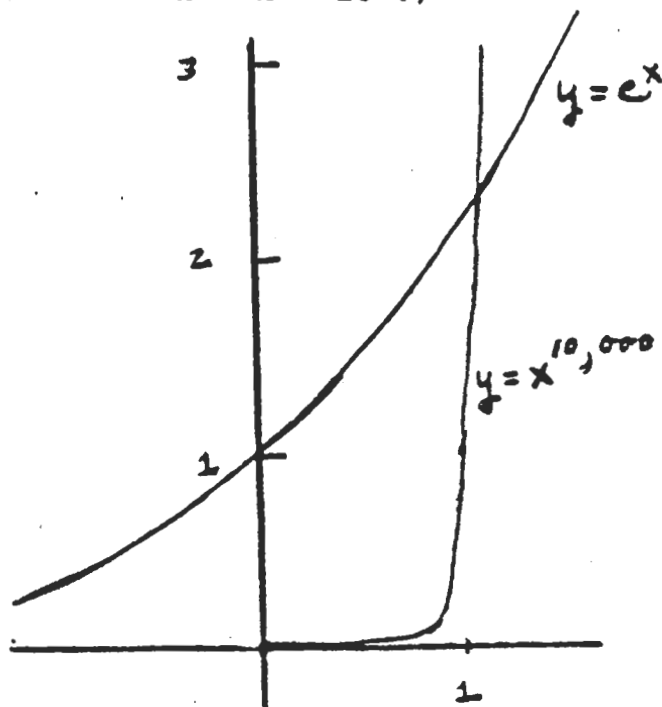
What does this theorem mean? Note that for any function  $f(x)$  that goes to  $\infty$  as  $x$  goes to  $\infty$ , a positive power of  $f(x)$ , say  $[f(x)]^a$ , goes to  $\infty$  even more rapidly if the power  $a$  is large, and to goes to  $\infty$  more slowly if  $a$  is small. This theorem says that no matter how high a power  $b$  you raise  $\log x$  to, and how small a power  $a$  you raise  $x$  to, the power of  $\log x$  will still go to  $\infty$  more slowly than the power of  $x$ . Similarly, any

power of  $e^x$ , no matter how small, will go to  $\infty$  faster than any power of  $x$ , no matter how large.

For example, even though for small values of  $x$ , the graphs of the functions  $\log x$  and  $x^{.0001}$  appear as in the accompanying figure, it is still true that eventually the function  $f(x) = x^{.0001}$  becomes much larger than  $\log x$ .



Similar graphs for the functions  $x^{10,000}$  and  $e^x$  can be obtained by exchanging the axes in this figure. Although  $x^{10,000}$  shoots up very rapidly to begin with, eventually  $e^x$  becomes much larger than  $x^{10,000}$ . (In fact, these curves cross again between  $x = 10^5$  and  $x = 10^6$ .)



L'Hopital's rule for  $\infty/\infty$ .

Theorem. Suppose  $f(x) \rightarrow \infty$  and  $g(x) \rightarrow \infty$  as  $x \rightarrow a$ . If

$$\frac{f'(x)}{g'(x)} \rightarrow L \text{ as } x \rightarrow a,$$

then also  $f(x)/g(x) \rightarrow L$  as  $x \rightarrow a$ .

This result holds whether a and L are finite or infinite, and it also holds if the limits are one-sided.

Proof. Case 1. We prove the theorem first in the case where a is finite and  $x \rightarrow a+$ .

The hypotheses of the theorem imply that f and g are defined and positive on some interval of the form  $(a, b]$ , and that  $f'$  and  $g'$  exist and  $g' \neq 0$  on some such interval.

Let  $x_0$  be a fixed point of this interval. (We shall specify how to choose  $x_0$  later.) Then let  $x$  be a point of this interval that is very close to a. Just how close will be determined later. For now we merely require that  $a < x < x_0$  and that  $f(x) > f(x_0)$  and  $g(x) > g(x_0)$ . (Since f and g go to  $\infty$  as  $x \rightarrow a+$ , these inequalities hold if x is close enough to a.) Then we compute.

Let us apply the Cauchy mean-value theorem to the interval  $[x, x_0]$ . We conclude that there is a c with  $x < c < x_0$  such that

$$f'(c)[g(x_0) - g(x)] = g'(c)[f(x_0) - f(x)]$$

or

$$f'(c)g(x)[g(x_0)/g(x) - 1] = g'(c)f(x)[f(x_0)/f(x) - 1]$$

or

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \left[ \frac{(g(x_0)/g(x)) - 1}{(f(x_0)/f(x)) - 1} \right].$$

For convenience, let  $\lambda(x)$  denote the expression in brackets; then

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \lambda(x).$$

Note that  $\lambda(x) \rightarrow 1$  as  $x \rightarrow a+$ .

Now we verify the theorem in the case where  $L$  is finite. By choosing  $x_0$  close to  $a$ , we can ensure that  $f'(c)/g'(c)$  is close to  $L$  (since  $a < c < x_0$ ); then we can make  $\lambda(x)$  close to 1 by requiring that  $x$  be very close to  $a$ . Then  $f(x)/g(x)$  will be close to  $L$ . The only question is: how close is "close enough"? Let us set

$$\epsilon_1 = |(f'(c)/g'(c)) - L| \quad \text{and} \quad \epsilon_2 = |\lambda(x) - 1|.$$

Then

$$(*) \quad \left| \frac{f'(c)}{g'(c)} \cdot \lambda(x) - L \right| = |(L \pm \epsilon_1)(1 \pm \epsilon_2) - L| \leq |\epsilon_1| + |L\epsilon_2| + |\epsilon_1\epsilon_2|.$$

This inequality tells us how to proceed. Suppose  $0 < \epsilon < 1$ . First, we choose  $x_0$  so that for all  $c$  with  $a < c < x_0$ , we have  $\epsilon_1 < \epsilon/3$ . Now  $x_0$  is fixed. Then choose  $\delta > 0$  so that for  $a < x < a + \delta$ , we have  $g(x) > g(x_0)$  and  $f(x) > f(x_0)$  and

$$|\lambda(x) - 1| = \epsilon_2 < \epsilon/3(1 + |L|).$$

Then for  $a < x < a + \delta$ , inequality (\*) tells us that

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} \lambda(x) - L \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon^2}{9} < \epsilon, \text{ as desired.}$$

Finally, we consider the case where  $L$  is infinite. Given  $M > 0$ , we want to show that  $f(x)/g(x) > M$  for  $x$  close to  $a$ . First, choose  $x_0$  so that for all  $c$  with  $a < c < x_0$ , we have  $f'(c)/g'(c) > 2M$ . Then choose  $\delta$  so that for  $a < x < a + \delta$ , we have  $g(x) > g(x_0)$  and  $f(x) > f(x_0)$  and  $\lambda(x) > 1/2$ . It follows that, for  $a < x < a + \delta$ ,

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \lambda(x) > 2M \cdot \frac{1}{2} = M.$$

We have now proved the rule in the case  $x \rightarrow a+$ . The case  $x \rightarrow a-$  follows readily, as we now show. Note that as  $x$  approaches  $a$  from the left,  $u = a - x$  approaches 0 from the right. Then

$$\begin{aligned} \lim_{x \rightarrow a-} (f(x)/g(x)) &= \lim_{u \rightarrow 0+} f(a-u)/g(a-u) \\ &= \lim_{u \rightarrow 0+} (-1)f'(a-u)/(-1)g'(a-u) \\ &= \lim_{x \rightarrow a-} f'(x)/g'(x), \end{aligned}$$

if the latter limit exists.

The case  $x \rightarrow a$ , with  $a$  finite, follows from the two cases  $x \rightarrow a+$  and  $x \rightarrow a-$ .

Finally, the case  $x \rightarrow \infty$  follows from the computation

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x)/g(x) &= \lim_{t \rightarrow 0+} f(1/t)/g(1/t) \\ &= \lim_{t \rightarrow 0+} (-1/t^2)f'(1/t)/(-1/t^2)g'(1/t) \\ &= \lim_{x \rightarrow \infty} f'(x)/g'(x), \end{aligned}$$

if the latter limit exists.  $\square$

### The behavior of log and exp

We now derive the theorem on p. P.2 from L'Hopital's rule. Consider first the log function. Given  $c > 0$ , we compute

$$\begin{aligned} \lim_{x \rightarrow \infty} (\log x)/x^c &= \lim_{x \rightarrow \infty} x^{-1}/cx^{c-1} \text{ by L'Hopital's rule} \\ &= \lim_{x \rightarrow \infty} 1/cx^c = 0. \end{aligned}$$

Then we set  $c = b/a$  and compute

$$\lim_{x \rightarrow \infty} (\log x)^a/x^b = \lim_{x \rightarrow \infty} [\log x/x^{b/a}]^a = 0,$$

as desired.

Now we consider the exp function. Given  $c > 0$ , we compute

$$\begin{aligned} \lim_{x \rightarrow \infty} e^{cx}/x &= \lim_{x \rightarrow \infty} ce^{cx}/1 \text{ by L'Hopital's rule} \\ &= \infty. \end{aligned}$$

Then we set  $c = a/b$  and compute

$$\lim_{x \rightarrow \infty} (e^x)^a/x^b = \lim_{x \rightarrow \infty} [e^{cx}/x]^b = \infty,$$

as desired.

Finally, we note that

$$\begin{aligned}\lim_{x \rightarrow 0^+} x^a \log x &= \lim_{t \rightarrow \infty} (1/t^a) \log(1/t) \\ &= \lim_{t \rightarrow \infty} \frac{-\log t}{t^a} = 0,\end{aligned}$$

as desired  $\square$

Example. Although

$$\lim_{x \rightarrow \infty} \frac{x + \sin x}{x}$$

assumes the indeterminate form  $\infty/\infty$ , L'Hopital's rule does not apply, since the function  $(1 + \cos x)/1$  oscillates rather than approaches a limit as  $x \rightarrow \infty$ . However,

$$\frac{x + \sin x}{x} = 1 + \frac{\sin x}{x},$$

which approaches 1 because  $|\sin x|/x \leq 1/x$  for  $x > 0$ .

This example shows that the converse of L'Hopital's rule is not true. For this is a case where  $f(x) \rightarrow \infty$  and  $g(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , and  $f(x)/g(x)$  approaches a limit, even though  $f'(x)/g'(x)$  does not approach a limit.