

A family of non-analytic functions.

Let  $m \geq 0$  be any nonnegative integer. Define

$$f_m(x) = \begin{cases} \frac{e^{-1/x^2}}{x^m} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

We will show that each of the functions  $f_m(x)$  has continuous derivatives of all orders, for all  $x$ . We also show that none of them is analytic near 0; that is, none of them equals a power series of the form  $\sum a_n x^n$  in an interval about 0.

Theorem 1. (a) The function  $f_m(x)$  is continuous for all  $x$ .

(b) Furthermore,  $f'_m(x)$  exists for all  $x$  and satisfies the equation

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x).$$

Proof. (a) The general theorem about composites of continuous functions shows that  $f_m(x)$  is continuous when  $x \neq 0$ . To prove continuity at  $x = 0$ , we must show that

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^m} = 0.$$

The substitution  $\mu = 1/x^2$  simplifies the calculation. We have

$$\lim_{x \rightarrow 0} \frac{e^{-1/x^2}}{x^m} = \lim_{\mu \rightarrow \infty} \frac{e^{-\mu}}{1/\mu^{m/2}} = \lim_{\mu \rightarrow \infty} \frac{\mu^{m/2}}{e^{\mu}}.$$

This limit is zero because  $e^\mu$  approaches infinity faster than any power of  $\mu$ , as  $\mu \rightarrow \infty$ .

(b) We check differentiability. If  $x \neq 0$ , we calculate directly:

$$\begin{aligned} f'_m(x) &= D\left(\frac{1}{x^m} e^{-1/x^2}\right) = \frac{-m}{x^{m+1}} e^{-1/x^2} + \frac{1}{x^m} e^{-1/x^2} \left(\frac{2}{x^3}\right) \\ &= -mf_{m+1}(x) + 2f_{m+3}(x). \end{aligned}$$

To show the derivative exists at  $x = 0$ , we apply the definition of the derivative:

$$\begin{aligned} f'_m(0) &= \lim_{h \rightarrow 0} \frac{f_m(0+h) - f_m(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(e^{-1/h^2}/h^m) - 0}{h} = \lim_{h \rightarrow 0} \frac{e^{-1/h^2}}{h^{m+1}} \end{aligned}$$

This limit is zero, by part (a). Therefore, the derivative exists at  $x = 0$  and equals 0. Thus the formula

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

holds when  $x = 0$ .

Theorem 2. The function  $f_m(x)$  has continuous derivatives of all orders, for all  $x$ , but  $f_m(x)$  does not equal a power series  $\sum a_n x^n$  on any interval about 0.

Proof. We know that each function  $f_m(x)$  is differentiable, for all  $x$ . The equation

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

shows us that  $f'_m(x)$  is differentiable, for each  $x$ . This is the same as saying that derivative  $f''_m(x)$  exists for all  $x$ .

In general, we proceed by induction. Suppose we are given that the  $n^{\text{th}}$  derivative of each function  $f_m(x)$  exists, for all  $x$ . Then the preceding equation shows that the  $n^{\text{th}}$  derivative of the function  $f'_m(x)$  also exists, for all  $x$ . This is the same as saying that the  $(n+1)^{\text{st}}$  derivative of  $f_m(x)$  exists.

It follows that the  $n^{\text{th}}$  derivative of  $f_m(x)$  exists, for all  $x$  and all  $n$ . And of course it is continuous because the  $(n+1)^{\text{st}}$  derivative exists.

Now we suppose  $f_m(x) = \sum a_n x^n$  on some non-trivial interval about  $x = 0$ , and derive a contradiction. If  $f_m(x)$  equals this power series, then the coefficients  $a_n$  must satisfy the equations

$$a_n = \frac{f_m^{(n)}(0)}{n!}$$

for all  $n$ . We know that  $f_m(x)$  vanishes when  $x = 0$ . Using the equation

$$f'_m(x) = -mf_{m+1}(x) + 2f_{m+3}(x)$$

repeatedly, we see that all the derivatives of  $f_m(x)$  also vanish at  $x = 0$ . Therefore  $a_n = 0$  for all  $n$ , so  $f_m(x)$

is identically zero in some interval about  $x = 0$ . But this is not true; indeed the function  $f_m(x)$  vanishes only for  $x = 0$ .