Rational exponents - an application of the intermediate-value theorem.

It is a consequence of the intermediate-value theorem, that, given a positive integer n and a real number $a \geq 0$, there is exactly one real number $b \geq 0$ such that

$$b^n = a.$$

We denote b by $\sqrt[n]{a}$, and call it the $n^{\underline{th}}$ root of a. (See Theorem 3.9, p. 145 of Apostol.)

It follows from the general theorem about continuity of inverses that the $n^{\underline{th}}$ root function, defined by the rule

$$f(x) = \sqrt[n]{x} \quad \text{for} \quad x \geq 0,$$

is continuous. (See Theorem 3.10, p. 147 of Apostol.)

Now (finally!) we can introduce rational exponents. We do so only when the base is a positive real number.

Definition. Let r be a rational number; let a be a positive real number. We can write $r = m/n$, where m and n are integers and n is positive. We then define

$$a^r = (\sqrt[n]{a})^m.$$

(Here we use the fact that $\sqrt[n]{a}$ is non-zero, so m can be negative.)

We must show that this definition makes sense. A problem might arise from the fact that the number r can be represented as a ratio of integers in many different ways. We must show that the value of a^r does not depend on how we represent r. This is the substance of the following lemma.

Lemma 1. *Suppose* $m/n = p/q$, *where* m, n, p, q *are integers, and* n *and* q *are positive. Then* $(\sqrt[n]{a})^m = (\sqrt[q]{a})^p$.

Proof. Let $c = \sqrt[n]{a}$ and $d = \sqrt[q]{a}$. Then $a = c^n$ and $a = d^q$ by definition. Because $m/n = p/q$, we have $mq = np$. Using these facts, we compute

$$a^p = (c^n)^p = c^{np} = c^{mq} = (c^m)^q, \quad \text{and}$$

$$a^p = (d^q)^p = d^{qp} = (d^p)^q, \quad \text{so that}$$

$$(c^m)^q = (d^p)^q.$$

(We use here the laws of integral exponents.) We conclude (by uniqueness of the q^{th} roots) that

$$c^m = d^p, \quad \text{or}$$

$$(\sqrt[n]{a})^m = (\sqrt[q]{a})^p. \quad \square$$

On the basis of Lemma 1, we know that a^r is well-defined if r is a rational number and a is positive. In particular, we have the equation

$$a^{1/n} = \sqrt[n]{a},$$

by definition. The definition of $a^{m/n}$ can then be written in the form

$$a^{m/n} = (a^{1/n})^m.$$

Consider now the three basic laws of exponents. We already know that these laws hold in the following cases:

(i) positive integral exponents; arbitrary bases.

(ii) integral exponents; non-zero bases.

We now comment that these laws also hold in the following case:

(iii) rational exponents; positive bases.

The proof is not difficult, but it is tedious. It is given in Theorem 2 following.

Later on, we shall extend our definition to arbitrary <u>real</u> exponents; that is, we shall define a^x when x is an arbitrary real number (and a is a positive real number). Furthermore, we shall verify that the laws of exponents also holds in this new situation; i.e., in the case:

(iv) real exponents; positive bases.

So you can skip the proof of Theorem 2 if you wish, for we are going to prove the more general result involving real exponents later on.

Before proving Theorem 2, we make the following remark about negative bases: If a is negative, one can still define $\sqrt[n]{a}$ provided n is odd. For in that case there exists exactly one real number b such that $b^n = a$. We shall define $\sqrt[n]{a} = b$ in this case. It is tempting to use exponent notation in this situation, defining $a^{m/n} = (\sqrt[n]{a})^m$ if n is odd and a is negative. However, this practice is dangerous! For the laws of exponents do not always hold in these circumstances. For example, if we used this definition, we would have

$$((-8)^2)^{1/6} = 2, \quad \text{while} \quad (-8)^{1/3} = -2.$$

Thus the second law of exponents would not hold in this situation. For this reason, we make the following convention:

We shall use rational exponent notation only when the base is positive.

Now we verify the laws of exponents for rational exponents and positive bases.

Theorem 2. If r and s are rational numbers, and if a and b are positive real numbers, then

(i) $a^r a^s = a^{r+s}$,

(ii) $(a^r)^s = a^{rs}$,

(iii) $a^r b^r = (ab)^r$.

Proof. Let $r = m/n$ and $s = p/q$, where m, n, p, q are integers, and where n and q are positive.

To prove (i), we note that

$$a^r a^s = a^{m/n}\, a^{p/q}$$

$$= a^{mq/nq}\, a^{np/nq}$$

$$= (\sqrt[nq]{a})^{mq}\, (\sqrt[nq]{a})^{np} \quad \text{by definition,}$$

$$= (\sqrt[nq]{a})^{mq+np} \quad \text{by (iii) for integral exponents,}$$

$$= a^{(mq+np)/nq} \quad \text{by definition,}$$

$$= a^{r+s}.$$

To prove (ii), we verify first that

$$(\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

Let $c = \sqrt[n]{a}$; then $c^n = a$ by definition. We compute

$$a^m = (c^n)^m = c^{nm} = (c^m)^n$$

by (ii) for integral exponents. By uniqueness of $n^{\underline{th}}$ roots, we have

$$\sqrt[n]{a^m} = c^m = (\sqrt[n]{a})^m,$$

as desired.

It now follows that

$$(*) \qquad a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}.$$

The first equation follows from the definition of $a^{m/n}$, and the second from what we just proved. The formula (*) is of course a special case of our desired formula (ii).

Now we prove (ii) in general: Let

$$c = (a^r)^s = (a^{m/n})^{p/q}.$$

Then

$$c = (((a^m)^{1/n})^p)^{1/q} \quad \text{by (*) (applied twice)}$$

$$= (((a^m)^p)^{1/n})^{1/q} \quad \text{by (*).}$$

It follows that

$$c^q = (((a^m)^p)^{1/n}, \quad \text{and}$$

$$(c^q)^n = (a^m)^p, \quad \text{by definition, so that}$$

$$c^{qn} = a^{mp} \quad \text{by (ii) for integral exponents.}$$

Then

$$c = \sqrt[qn]{a^{mp}} \quad \text{by definition,}$$

$$= (a^{mp})^{1/nq} \quad \text{by definition,}$$

$$= a^{mp/nq} \quad \text{by (*),}$$

$$= a^{rs}.$$

To check (iii), let $c = \sqrt[n]{a}$ and $d = \sqrt[n]{b}$. We first note that

$$(cd)^n = c^n d^n \quad \text{by (iii) for integral exponents,}$$

$$= ab \quad \text{by definition.}$$

It follows that

$$cd = \sqrt[n]{ab}.$$

We then prove (iii) as follows:

$$a^{m/n}b^{m/n} = (\sqrt[n]{a})^m(\sqrt[n]{b})^m = c^m d^m \quad \text{by definition,}$$

$$= (cd)^m \quad \text{by (iii) for integral exponents,}$$

$$= (\sqrt[n]{ab})^m \quad \text{by (*),}$$

$$= (ab)^{m/n} \quad \text{by definition.}$$

Thus the three laws hold for rational exponents. □