

IntegrationThe substitution rule

Apostol proves only one version of the substitution rule, the one given in Theorem 1 following. Sometimes the converse is needed; we prove this result in Theorem 2.

Theorem 1. Assume that $f(u)$ and $g(x)$ and $g'(x)$ are continuous, and that $f(g(x))$ is defined for all x in the domain of g . If

$$\int f(u)du = P(u) + C, \text{ then}$$

$$\int f(g(x))g'(x)dx = P(g(x)) + C.$$

Proof. We are given that $P'(u) = f(u)$. The chain rule implies that the derivative of $P(g(x))$ equals

$$P'(g(x))g'(x) = f(g(x))g'(x).$$

This is just the desired result. \square

Theorem 2. (Partial converse) Assume that $f(u)$ and $g(x)$ and $g'(x)$ are continuous, and that $f(g(x))$ is defined for all x in the domain of g . Assume also that $u = g(x)$ has the differentiable inverse function $x = h(u)$. If

$$(*) \quad \int f(g(x))g'(x)dx = Q(x) + C, \text{ then}$$

$$(**) \quad \int f(u)du = Q(h(u)) + C.$$

Proof. Applying Theorem 1, we can substitute $x = h(u)$ in the given formula (*) to obtain the equation

$$\int [f(g(h(u)))g'(h(u))]h'(u)du = Q(h(u)) + C.$$

Because h is the inverse function to g , we know that $g(h(u)) = u$ and

$$h'(u) = 1/g'(h(u)).$$

This formula thus takes the form

$$\int f(u)du = Q(h(u)) + C.$$

which is the equation we wished to prove. \square

Example 1. The usual application of the substitution rule uses Theorem 1. One begins with the given integrand, and tries to write it in the form $f(g(x))g'(x)$ for some suitable function f and g , where f is a function we know how to integrate. For example, suppose we wish to compute the integral

$$\int x^2 \cos(x^3) dx.$$

We see this is almost of the form $\int \cos u \, du$ if we set $u = x^3$. That is, we "group x^2 with dx and supply a factor of 3", writing the integral in the form

$$\frac{1}{3} \int \cos(x^3) [3x^2 dx].$$

Then because we know that $\int \cos u \, du = \sin u + C$, we conclude from Theorem 1 that our given integral equals

$$\frac{1}{3} \sin(x^3) + C.$$

Example 2. On the other hand, sometimes Theorem 2 is the one that is useful. It often applies when there is nothing obvious to "group with the dx " to simplify the integrand. Trigonometric substitutions are of this type.

For example, consider the integral $\int f(u) du$, where

$$f(u) = \frac{1}{\sqrt{1+u^2}}.$$

This is not something we know how to integrate. However, the substitution $u = \tan x$ *will* simplify the expression $\sqrt{1+u^2}$ at least. It is an acceptable substitution, since it has the differentiable inverse function $x = \arctan u$

Using this substitution, we have

$$1 + u^2 = 1 + \tan^2 x = \sec^2 x.$$

Then

$$\sqrt{1+u^2} = \sec x;$$

the sign is +, because x lies between $-\pi/2$ and $+\pi/2$, so $\sec x = 1/\cos x$ is positive.

And of course we have

$$1 + u^2 = \sec^2 x.$$

Hence the integral

$$\int (1/\sqrt{1+u^2}) du$$

takes the form

$$\int (1/\sec x) \sec^2 x dx$$

which we know how to integrate. Indeed,

$$\int \sec x dx = \log|\sec x + \tan x| + C.$$

Then we can apply Theorem 2 to conclude that

$$\int \left[1/\sqrt{1+u^2} \right] du = \log|\sec(\arctan u) + \tan(\arctan u)| + C.$$

This answer can be written more simply. For if $x = \arctan u$, then $\sec x = \sqrt{1+u^2}$, as noted earlier, and $\tan x = u$. Hence we have the formula

$$\int \left[1/\sqrt{1+u^2} \right] du = \log|\sqrt{1+u^2} + u| + C.$$

A strategy for integration.

Step 1. Determine whether you can simplify the integrand easily, using algebraic manipulations (such as completing the square), trig identities, or a simplifying substitution (especially for the "inside function" in $f(g(x))$.)

Step 2. Examine the form of the integrand to determine the appropriate method.

(a) A product of two dissimilar functions suggests integration by parts. [Examples: $x^2 \sin x$, xe^x .] The same holds for a function whose derivative is a more familiar function than the function itself. [Examples: $\log x$, $\arctan x$.]

(b) Rational functions of x can always be integrated by the method of partial fractions.

(c) Powers of trig functions can be integrated using the half-angle formulas, various substitutions or (if necessary) reduction formulas.

(d) For integrands involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$, a trig substitution is often helpful.

(e) [Optional: Any rational function of $\sin x$ and $\cos x$ can be reduced to a rational function of u by means of the substitution $u = \tan(x/2)$.]

Exercises

Evaluate the following:

1. $\int \cos^2 x \, dx.$

(Use either a half-angle formula derived from the identity

$$\cos 2x = 1 - 2 \sin^2 x = 2 \cos^2 x - 1$$

or the reduction formula of p. 221.)

2. $\int_0^{1/2} \sqrt{1 - \mu^2} \, d\mu.$

3. Integrate $\int \frac{d\mu}{1 - \mu^2}$ by expressing $\frac{1}{1 - \mu^2}$ in the

form $\frac{a}{1 - \mu} + \frac{b}{1 + \mu}$. Use this formula to evaluate

$$\int \sec x \, dx = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{\cos x \, dx}{1 - \sin^2 x}.$$

4. $\int \frac{d\mu}{\sqrt{\mu^2 - 1}}.$

5. $\int_0^1 \frac{d\mu}{(\sqrt{\mu^2 + 1})^3}.$

6. Compute

$$\int_0^1 x f''(2x) \, dx,$$

given that f'' is continuous for all x , and

$$f(0) = 1, \quad f'(0) = 3.$$

$$f(1) = 5, \quad f'(1) = 2.$$

$$f(2) = 7, \quad f'(2) = 4.$$

7. Evaluate $\int \frac{x^4 + 2}{x^4 + x^3 + x^2} \, dx$ completely, including the constants

A, B, ... in the partial fraction decomposition.